Minimal degenerations for quiver varieties

jt with Travis Schedler.

1) The nilpotent cone
2) Nakajima quiver varieties.
3) Minimal degenerations.
4) Examples

1) The nilpotent cone.

Let $G = GL_n(\mathbb{C})$ act on $\mathfrak{g} = Mat_{n\times n}(\mathbb{C})$

by conjugation: $g \cdot X := gXg^{-1}$.

$\mathcal{N} = \{ X \in \mathfrak{g} \mid X^n = 0 \}$ nilpotent cone.

Recall that, up to conjugation, X is classified

by its Jordan blocks:

$$X \sim \begin{pmatrix} J_{p_1} & & & \\ & J_{p_2} & & \\ & & \ddots & \\ & & & J_{p_k} \end{pmatrix}$$ where J_p is the Jordan block of size p.

We can assume $p_1 \geq p_2 \geq \cdots$ so $(p_1, \dots, p_k) \vdash n$

Hence $\mathcal{N} = \coprod_{\lambda \vdash n} \mathcal{O}_\lambda$ is a finite union of G-orbits.

We say that $\lambda \geq \mu$ if $\lambda_1 + \cdots + \lambda_r \geq \mu_1 + \cdots + \mu_r$ $\forall\, r \geq 1$

(dominance ordering)

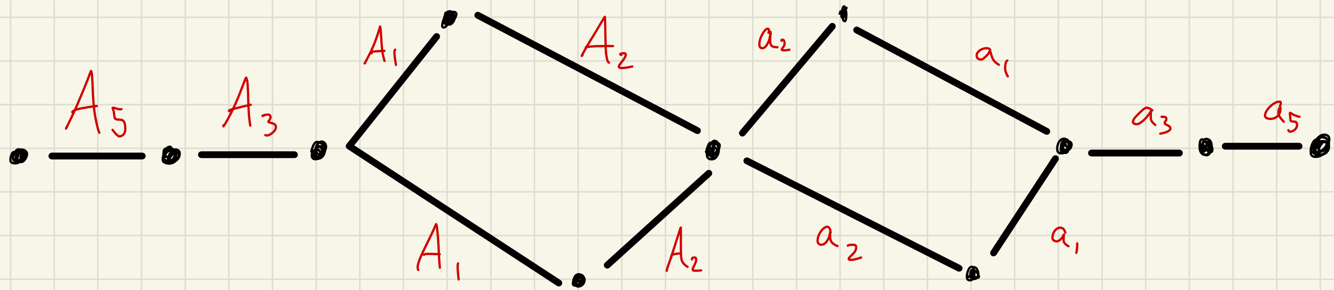
e.g. $(n) \geq \mu$ for all $\mu \vdash n$.

Then $\mathcal{O}_\mu \subset \overline{\mathcal{O}_\lambda}$ if and only if $\lambda \geq \mu$.

Write $\lambda - \mu$ if $\lambda > \mu$ and $\nexists\, \nu :$ $\lambda > \nu > \mu$.

$\rightsquigarrow$ Hasse diagram of closure ordering.

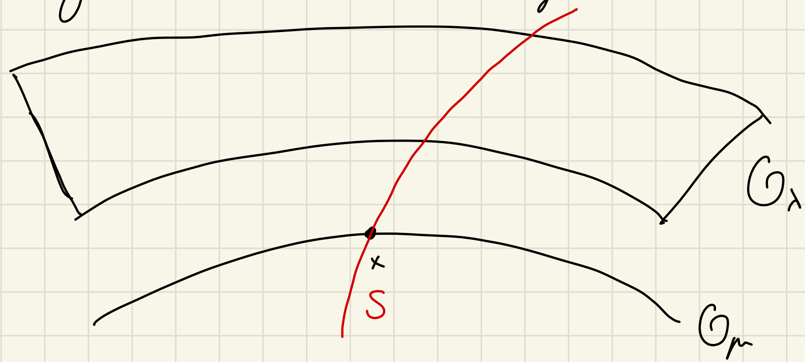
$n = 6$



λ	(6)	$(5,1)$	$(4,2)$	$(4,1,1)$	$(3,3)$	$(3,2,1)$	$(2,2,2)$	$(3,1,1,1)$	$(2,2,1,1)$	$(2,1^4)$	(1^6)
$\dim \mathcal{O}_\lambda$	30	28	26	24	24	22	18	18	16	10	0

If $\lambda - \mu$ and $x \in \mathcal{O}_\mu$ then take a slice $x \in S \subseteq \mathfrak{g}$ transverse to $(\mathcal{O}_\mu, x)$ and consider $\overline{\mathcal{O}_\lambda} \cap S = S_{\lambda\mu}$.

- isolated singularity called minimal degeneration



Examples 1) $\mathcal{O}_{(n)} = \mathcal{O}_{reg}$, $\mathcal{O}_\mu = \mathcal{O}_{(n-1,1)} = \mathcal{O}_{subreg}$

$$S_{(n),(n-1,1)} \cong \mathbb{C}^2/(\mathbb{Z}/n\mathbb{Z}) \quad : \quad A_{n-1}$$

2) $\mathcal{O}_{(2,1^{n-2})} = \mathcal{O}_{min}$ $\qquad$ $\mathcal{O}_{\mu} = \mathcal{O}_{(1^n)} = \{0\}$.

$$S_{(2,1^{n-2}),(1^n)} = \overline{\mathcal{O}_{min}} = \{X \mid rk X \leq 1,\ X^2=0\} \quad : a_{n-1}$$

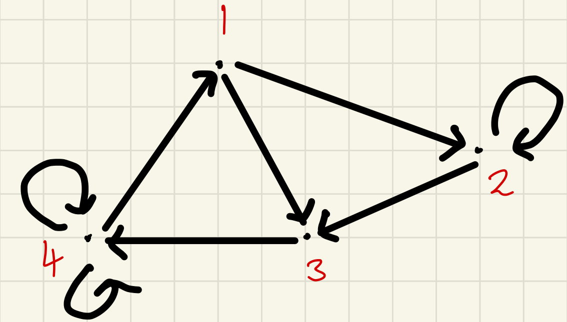
$$T^*\mathbb{P}^{n-1} \longrightarrow \overline{\mathcal{O}_{min}} \qquad \text{collapsing zero section.}$$

<u>Theorem (Kraft–Procesi)</u>

Each minimal degeneration in $\mathcal{N}$ is isomorphic to a_k or A_k some $1 \leq k \leq n-1$.

Nakajima quiver varieties

$Q = (Q_0, Q_1)$ finite quiver



$\rightsquigarrow \Pi(Q)$ preprojective algebra

$Q_0 = \{1, 2, 3, 4\}$

$|Q_1| = 8.$

a quotient of $\mathbb{C}\overline{Q}$

$\uparrow$ doubled quiver.

For a dimension vector $d \in \mathbb{N}^{|Q_0|}$

get

$\mathcal{M}(Q,d)$ = (coarse) moduli space of rep^n of $\Pi(Q)$ of dimension d.

closed point $\longleftrightarrow$ iso classes of semi-simple rep^n of dim d.

Symplectic leaves

$\mathcal{M}(Q,d)$ admits a finite stratification by symplectic leaves.

leaves are rep^n type strata:

let $\Sigma = \{ \alpha \in \mathbb{N}^{|Q_0|} \mid \exists$ simple $\Pi(Q)$-module of dim $\alpha \}$.

Defn A Σ-coloured partition of d is $\tau : \Sigma \longrightarrow \mathcal{P}$ all partitions.

such that $d = \sum_{\alpha} |\tau(\alpha)| \alpha$.

Theorem (Crawley-Boevey) $\mathcal{M}(Q,d) = \bigsqcup_{\tau \vdash d} \mathcal{M}(Q,d)_\tau$

each $\mathcal{M}(Q,d)_\tau$ is smooth connected.

If M is semi-simple then

$$M = \left(M_{11}^{\oplus p_{11}} \oplus M_{12}^{\oplus p_{12}} \oplus \cdots \right) \oplus \left(M_{21}^{\oplus p_{21}} \oplus M_{22}^{\oplus p_{22}} \oplus \cdots \right) \oplus \cdots$$

where $\dim M_{11} = \dim M_{12} = \cdots = \beta^{(1)}$ $\qquad$ $M_{11} \not\cong M_{12} \not\cong \cdots\cdots$

$\dim M_{21} = \dim M_{22} = \cdots = \beta^{(2)}$ $\in \Sigma$

and $p_{i1} \geq p_{i2} \geq \cdots$ a partition.

$\rightsquigarrow$ τ a Σ- coloured partition.

$\overline{M_\tau}$ a union of strata M_ℓ

$\rightsquigarrow$ $\tau \geq \ell$ if $\overline{M_\tau} \supset M_\ell$.

Gives rise to Hesse diagram as before.

Theorem (Bellamy–Schedler)

The minimal degenerations in $\mathcal{M}(Q,d)$ are:

1) Kleinian singularities $\mathbb{C}^2/\Gamma$, $\Gamma \subseteq SL(2,\mathbb{C})$ finite group $\quad A_n, D_n, E_6, E_7, E_8.$

2) $\overline{\mathcal{O}}_{min}(\mathfrak{gl}_n)$ $\quad \mathfrak{a}_{n-1}$

3) $\mathbb{C}^{2g}/(\mathbb{Z}/2\mathbb{Z}) = \overline{\mathcal{O}}_{min}(\mathfrak{sp}_{2g})$ $\quad \mathfrak{c}_g$

4) $\mathrm{Spec}\,(\mathbb{C}[x_1 \cdots x_{2g}]_{\geq 2} \oplus \mathbb{C})$ $\quad m_g$ $\quad$ (normalization $\cong \mathbb{A}^{2g}$)

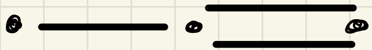
Key lemma

The following are all minimal imaginary roots for Q:

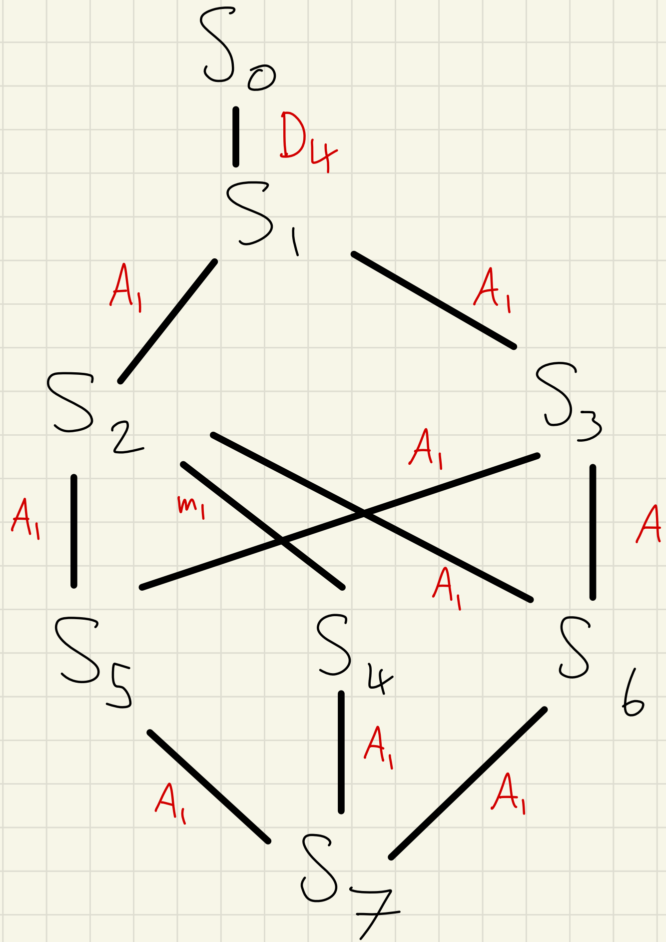
1) minimal imaginary root for affine Dynkin subquiver.
2) $(1,1)$ for n-Kronecker subquiver.
3) (1) for vertex with at least one loop.

Direct analogue to Ostrik-Malkin-Vybornov result.

Q



$d = (2, 4, 3)$



$\dim S_i$

8

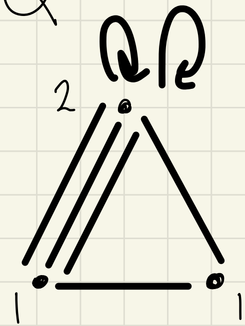
6

4

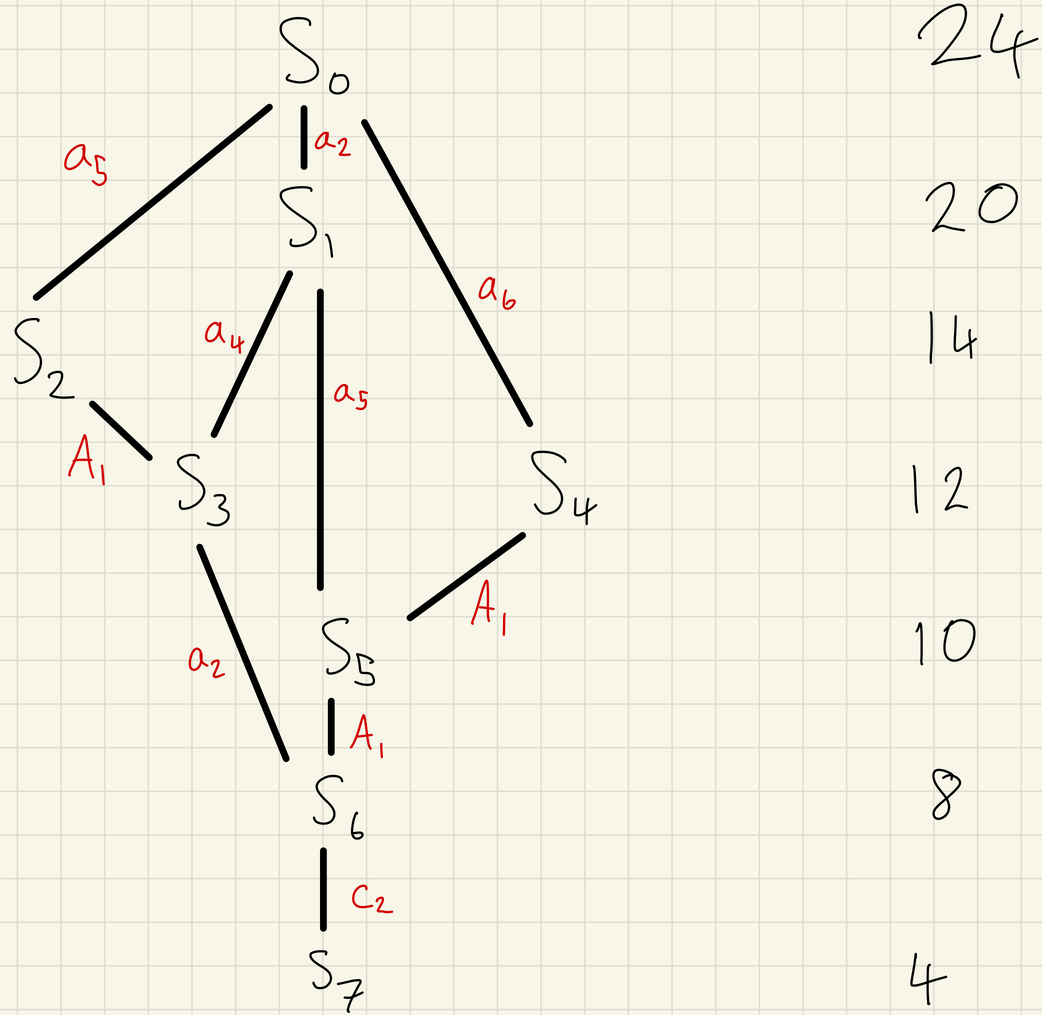
2

0

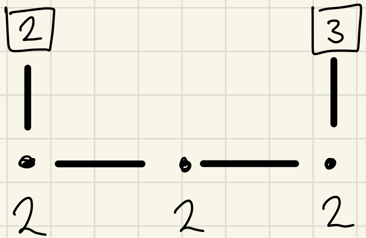
Q

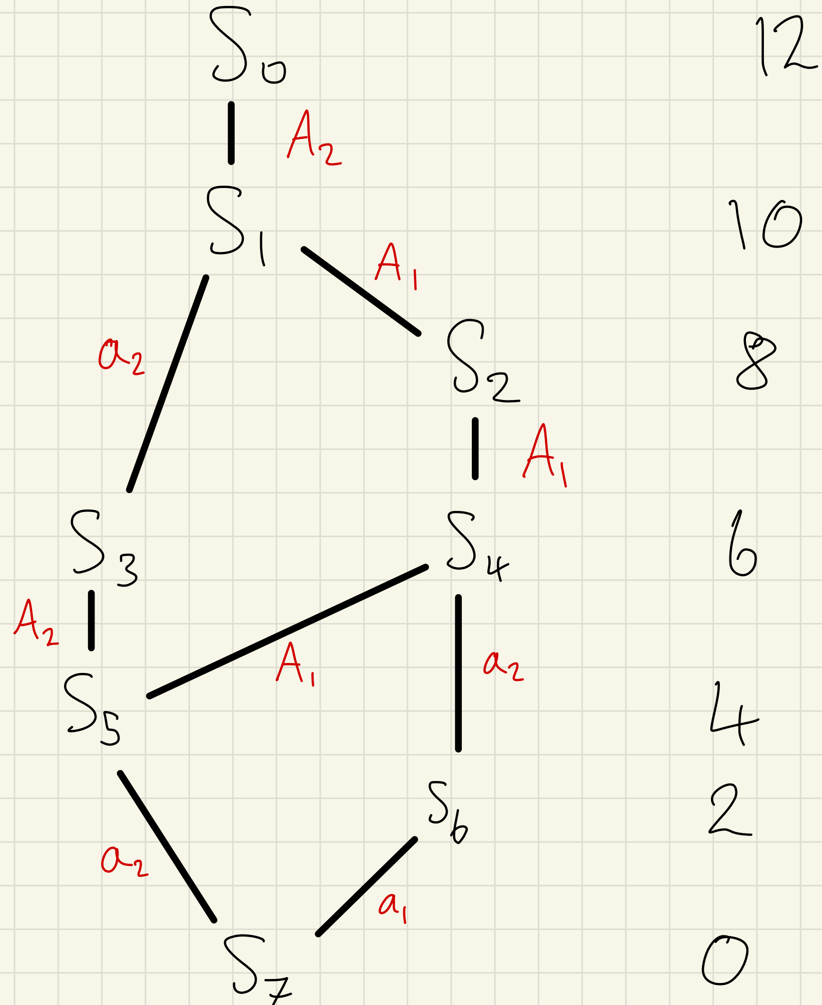

2
1
1

$d = (2, 1, 1)$

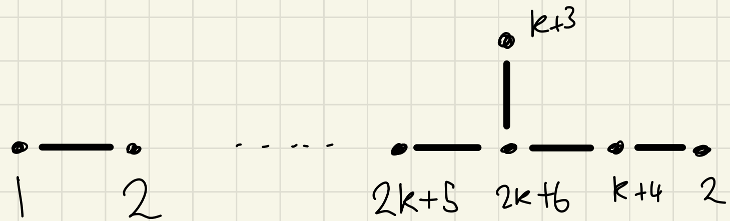

S_0
a_5
a_2
a_6
S_1
S_2
a_4
a_5
A_1
S_3
S_4
A_1
a_2
S_5
A_1
S_6
c_2
S_7
24
20
14
12
10
8
4

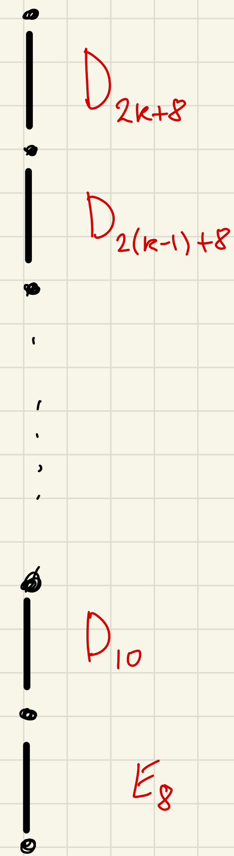
Q





(Hanany)

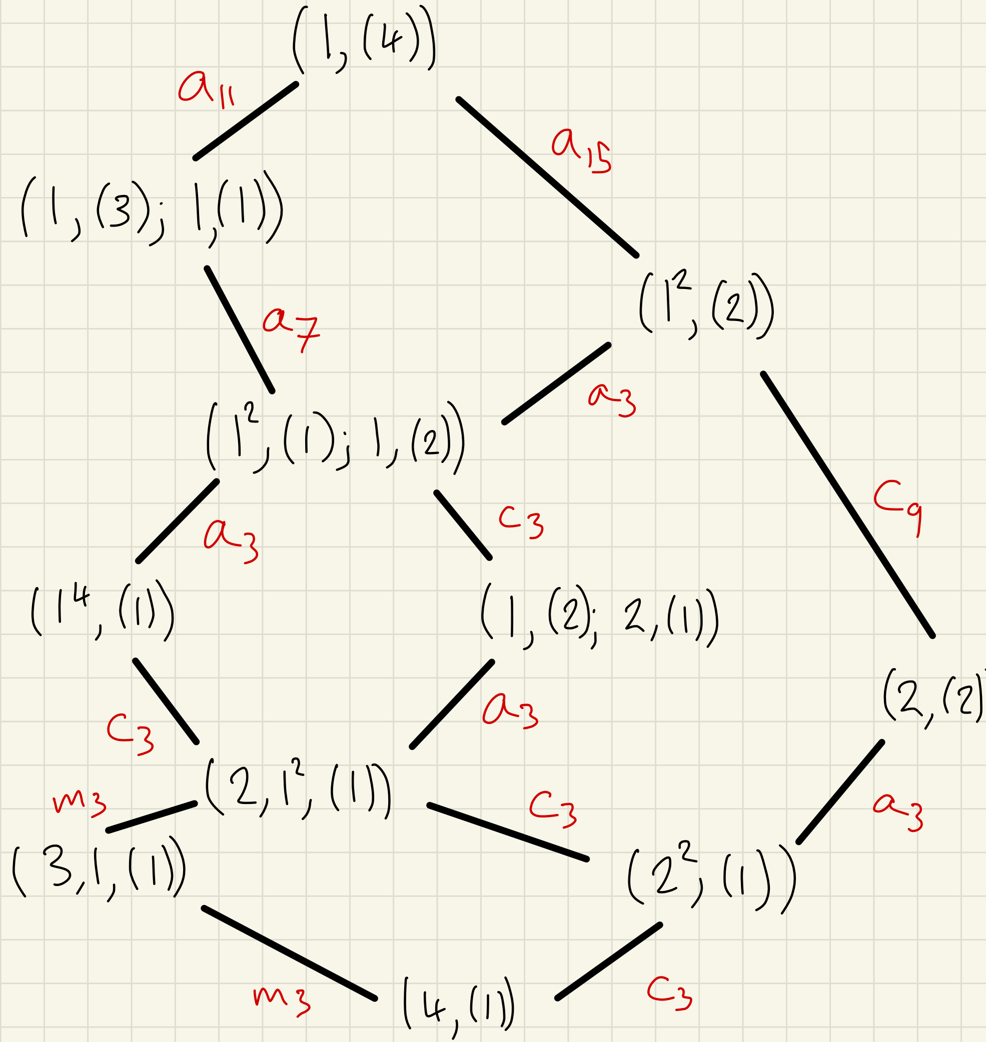




Q

3 loops

4



$(1,(4))$
a_{11}
a_{15}
$(1,(3);1,(1))$
$(1^2,(2))$
a_7
a_3
$(1^2,(1);1,(2))$
c_9
a_3
c_3
$(1^4,(1))$
$(1,(2);2,(1))$
$(2,(2))$
c_3
a_3
$(2,1^2,(1))$
m_3
c_3
a_3
$(3,1,(1))$
$(2^2,(1))$
m_3
c_3
$(4,(1))$

Normality of leaf closures

Classification shows that many leaf closures are <u>not</u> normal.

- Easy to describe normalization:

S_n = symmetric group.

$$\prod_{\beta \in \Sigma} \mathcal{M}(Q,\beta)^{\times \ell(\tau(\beta))} \Big/ S(\tau(\beta)) \longrightarrow \overline{S_\tau},$$

where $S(\lambda) = \prod_{i \geq 1} S_{n_i}$ if $\lambda = (1^{n_1}, 2^{n_2}, 3^{n_3} \ldots\ldots) \in \mathcal{P}$.

We say that imaginary roots $\alpha, \beta \in \Sigma$ have <u>real intersection</u>

if $\gamma \in \Sigma$, $\gamma < \alpha, \beta \implies \gamma$ is real.

<u>Theorem (B-S)</u>

If $\overline{S}_\tau$ is normal then

- each $\tau(\beta)$ is a rectangle for β imaginary.
- For $\alpha, \beta \in \Sigma_0$ imaginary, $\tau(\alpha), \tau(\beta) \neq \phi \implies$ α, β have real intersection.