Component groups of the stabilizers of nilpotent orbit representatives

Emanuele Di Bella, Willem A. De Graaf

UNIVERSITÀ DI TRENTO

Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$.

Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$.

Definition

An element $x \in \mathfrak{g}$ is said to be nilpotent if adx is a nilpotent endomorphism.

Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$.

Definition

An element $x \in \mathfrak{g}$ is said to be nilpotent if adx is a nilpotent endomorphism.

Let $G = Aut(\mathfrak{g})^0$. Then G is an algebraic subgroup of $\mathrm{GL}(\mathfrak{g})$ and its Lie algebra is $ad\mathfrak{g}$.

Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$.

Definition

An element $x \in \mathfrak{g}$ is said to be nilpotent if adx is a nilpotent endomorphism.

Let $G = Aut(\mathfrak{g})^0$. Then G is an algebraic subgroup of $\mathrm{GL}(\mathfrak{g})$ and its Lie algebra is $ad\mathfrak{g}$.

Definition

If e is a nilpotent element of $\mathfrak{g}$ then all the elements of the orbit $G \cdot e$ are nilpotent. In this case, the orbit is said to be nilpotent.

Theorem (Jacobson-Morozov)

For a nilpotent $e \in \mathfrak{g}$ there are h, f such that

$$[h, e] = 2e, [h, f] = -2f, [e, f] = h.$$

The triple (h, e, f) is called an $\mathfrak{sl}_2$-triple.

Theorem (Jacobson-Morozov)

For a nilpotent $e \in \mathfrak{g}$ there are h, f such that

$$[h, e] = 2e, [h, f] = -2f, [e, f] = h.$$

The triple (h, e, f) is called an $\mathfrak{sl}_2$-triple.

Definition

$Z_G(e) := \{g \in G \mid g(e) = e\}$.
$Z_G(h, e, f) := \{g \in G \mid g(h) = h, g(e) = e, g(f) = f\}$.

Theorem (Jacobson-Morozov)

For a nilpotent $e \in \mathfrak{g}$ there are h, f such that

$$[h, e] = 2e, [h, f] = -2f, [e, f] = h.$$

The triple (h, e, f) is called an $\mathfrak{sl}_2$-triple.

Definition

$Z_G(e) := \{g \in G \mid g(e) = e\}$.
$Z_G(h, e, f) := \{g \in G \mid g(h) = h, g(e) = e, g(f) = f\}$.

The aim of our work is to compute $Z_G(e)/Z_G^0(e)$, where e is the representative of a nilpotent orbit.

Theorem (Jacobson-Morozov)

For a nilpotent $e \in \mathfrak{g}$ there are h, f such that

$$[h, e] = 2e, [h, f] = -2f, [e, f] = h.$$

The triple (h, e, f) is called an $\mathfrak{sl}_2$-triple.

Definition

$Z_G(e) := \{g \in G \mid g(e) = e\}$.
$Z_G(h, e, f) := \{g \in G \mid g(h) = h, g(e) = e, g(f) = f\}$.

The aim of our work is to compute $Z_G(e)/Z_G^0(e)$, where e is the representative of a nilpotent orbit.

Theorem

Representatives of the component group of $Z_G(h, e, f)$ are also representatives of the component group of $Z_G(e)$.

Fix a base $\Delta = \{\alpha_1, \ldots, \alpha_l\}$ of the root system Φ of $\mathfrak{g}$ and a canonical generating set $\{h_i, x_{\pm\alpha_i}\}$.
Let π be a permutation of $\{1, \ldots, l\}$ such that $\langle \alpha_i, \alpha_j \rangle = \langle \alpha_{\pi(i)}, \alpha_{\pi(j)} \rangle$. It follows that there is a unique automorphism σ_π of $\mathfrak{g}$ such that $\sigma_\pi(h_i) = h_{\pi(i)}$, $\sigma_\pi(x_{\pm\alpha_i}) = x_{\pm\alpha_{\pi(i)}}$.

Fix a base $\Delta = \{\alpha_1, \dots, \alpha_l\}$ of the root system Φ of $\mathfrak{g}$ and a canonical generating set $\{h_i, x_{\pm\alpha_i}\}$.
Let π be a permutation of $\{1, \dots, l\}$ such that $\langle \alpha_i, \alpha_j \rangle = \langle \alpha_{\pi(i)}, \alpha_{\pi(j)} \rangle$. It follows that there is a unique automorphism σ_π of $\mathfrak{g}$ such that $\sigma_\pi(h_i) = h_{\pi(i)}$, $\sigma_\pi(x_{\pm\alpha_i}) = x_{\pm\alpha_{\pi(i)}}$.

Definition

The automorphism σ_π, constructed as above, is called diagram automorphism.
Since $\sigma_{\pi_1\pi_2} = \sigma_{\pi_1}\sigma_{\pi_2}$, we can define a (finite) group of diagram automorphisms of $\mathfrak{g}$ denoted Γ.

Proposition

With the notation above, we have the following:

$$Aut(\mathfrak{g}) = G \rtimes \Gamma.$$

Classical Lie Algebras

- $A_n := \mathfrak{sl}(n+1) = \{x \in \mathfrak{gl}(n+1) : \ \mathrm{tr}\ (x) = 0\}$
- $B_n := \mathfrak{so}(2n{+}1) = \left\{ x \in \mathfrak{gl}(2n+1) : sx = -sx^T, s = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & I_n \\ 0 & I_n & 0 \end{pmatrix} \right\}$
- $C_n := \mathfrak{sp}(2n) = \left\{ x \in \mathfrak{gl}(2n) : sx = -sx^T, s = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \right\}$
- $D_n := \mathfrak{so}(2n) = \left\{ x \in \mathfrak{gl}(2n) : sx = -sx^T, s = \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix} \right\}$

For the above Lie algebras we can compute the component group explicitly by constructing an algorithm which follows the theoretical approach given by J. C. Jantzen (2004).

Let $\hat{G} = O(V)$ and $\mathfrak{g} = \mathfrak{so}(V)$.
Let $\mathfrak{a}$ be the subalgebra spanned by an $\mathfrak{sl}_2$-triple in $\mathfrak{g}$, say (h, e, f). Then $V = V_1 \oplus \cdots \oplus V_m$, where V_i is an irreducible $\mathfrak{a}$-module of dimension d_i.

Let $\hat{G} = O(V)$ and $\mathfrak{g} = \mathfrak{so}(V)$.

Let $\mathfrak{a}$ be the subalgebra spanned by an $\mathfrak{sl}_2$-triple in $\mathfrak{g}$, say (h, e, f). Then $V = V_1 \oplus \cdots \oplus V_m$, where V_i is an irreducible $\mathfrak{a}$-module of dimension d_i.

From $\mathfrak{sl}_2$ theory, each V_i has a unique element v_i such that:

1. $f \cdot v_i = 0$;
2. $h \cdot v_i = (-d_i + 1)v_i$;
3. $e^{d_i} \cdot v_i = 0$ and $e^k \cdot v_i$ for $0 \leq k \leq d_i - 1$ is a basis of V_i.

Let $\hat{G} = O(V)$ and $\mathfrak{g} = \mathfrak{so}(V)$.
Let $\mathfrak{a}$ be the subalgebra spanned by an $\mathfrak{sl}_2$-triple in $\mathfrak{g}$, say (h, e, f). Then $V = V_1 \oplus \cdots \oplus V_m$, where V_i is an irreducible $\mathfrak{a}$-module of dimension d_i.
From $\mathfrak{sl}_2$ theory, each V_i has a unique element v_i such that:

1. $f \cdot v_i = 0$;
2. $h \cdot v_i = (-d_i + 1)v_i$;
3. $e^{d_i} \cdot v_i = 0$ and $e^k \cdot v_i$ for $0 \leq k \leq d_i - 1$ is a basis of V_i.

Let $M_s := span\{v_j : d_j = s\}$.

Let $\hat{G} = O(V)$ and $\mathfrak{g} = \mathfrak{so}(V)$.
Let $\mathfrak{a}$ be the subalgebra spanned by an $\mathfrak{sl}_2$-triple in $\mathfrak{g}$, say (h, e, f). Then $V = V_1 \oplus \cdots \oplus V_m$, where V_i is an irreducible $\mathfrak{a}$-module of dimension d_i.
From $\mathfrak{sl}_2$ theory, each V_i has a unique element v_i such that:

1. $f \cdot v_i = 0$;
2. $h \cdot v_i = (-d_i + 1)v_i$;
3. $e^{d_i} \cdot v_i = 0$ and $e^k \cdot v_i$ for $0 \leq k \leq d_i - 1$ is a basis of V_i.

Let $M_s := span\{v_j : d_j = s\}$.

Theorem

$$Z_{\hat{G}}(h, e, f) \quad \longrightarrow \quad \prod_{s\ odd} O(M_s) \times \prod_{s\ even} Sp(M_s)$$

is an isomorphism of groups.

For the explicit computation:

- For any odd s, find an element $g_s \in O(M_s)$ such that $det(g_s) = -1$;

Theorem

$$Z_{\hat{G}}(h, e, f) \quad \longrightarrow \quad \prod_{s\ odd} O(M_s) \times \prod_{s\ even} Sp(M_s)$$

is an isomorphism of groups.

For the explicit computation:

- For any odd s, find an element $g_s \in O(M_s)$ such that $det(g_s) = -1$;
- Compute the preimage $\hat{g}_s := 1 \times \cdots \times 1 \times g_s \times 1 \times \cdots \times 1$ for any odd s;

Theorem

$$Z_{\hat{G}}(h, e, f) \quad \longrightarrow \quad \prod_{s\ odd} O(M_s) \times \prod_{s\ even} Sp(M_s)$$

is an isomorphism of groups.

For the explicit computation:

- For any odd s, find an element $g_s \in O(M_s)$ such that $det(g_s) = -1$;
- Compute the preimage $\hat{g}_s := 1 \times \cdots \times 1 \times g_s \times 1 \times \cdots \times 1$ for any odd s;
- Let M be the group generated by the $\hat{g}_s$;

Theorem

$$Z_{\hat{G}}(h,e,f) \quad \longrightarrow \quad \prod_{s\ odd} O(M_s) \times \prod_{s\ even} Sp(M_s)$$

is an isomorphism of groups.

For the explicit computation:

- For any odd s, find an element $g_s \in O(M_s)$ such that $det(g_s) = -1$;
- Compute the preimage $\hat{g}_s := 1 \times \cdots \times 1 \times g_s \times 1 \times \cdots \times 1$ for any odd s;
- Let M be the group generated by the $\hat{g}_s$;
- For $g \in M$, compute $\sigma_g \in Z_G(h, e, f)$ as $\sigma_g(x) = gxg^{-1}$;

Theorem

$$Z_{\hat{G}}(h, e, f) \longrightarrow \prod_{s\ odd} O(M_s) \times \prod_{s\ even} Sp(M_s)$$

is an isomorphism of groups.

For the explicit computation:

- For any odd s, find an element $g_s \in O(M_s)$ such that $det(g_s) = -1$;
- Compute the preimage $\hat{g}_s := 1 \times \cdots \times 1 \times g_s \times 1 \times \cdots \times 1$ for any odd s;
- Let M be the group generated by the $\hat{g}_s$;
- For $g \in M$, compute $\sigma_g \in Z_G(h, e, f)$ as $\sigma_g(x) = gxg^{-1}$;
- Check whether $\sigma_g \in Z_G^0(h, e, f)$ to determine $Z_G(h, e, f)/Z_G^0(h, e, f)$.

Theorem

$$Z_{\hat{G}}(h, e, f) \quad \longrightarrow \quad \prod_{s\ odd} O(M_s) \times \prod_{s\ even} Sp(M_s)$$

is an isomorphism of groups.

For the explicit computation:

- For any odd s, find an element $g_s \in O(M_s)$ such that $det(g_s) = -1$;
- Compute the preimage $\hat{g}_s := 1 \times \cdots \times 1 \times g_s \times 1 \times \cdots \times 1$ for any odd s;
- Let M be the group generated by the $\hat{g}_s$;
- For $g \in M$, compute $\sigma_g \in Z_G(h, e, f)$ as $\sigma_g(x) = gxg^{-1}$;
- Check whether $\sigma_g \in Z_G^0(h, e, f)$ to determine $Z_G(h, e, f)/Z_G^0(h, e, f)$.
- In general, the component group is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^a$, where a is the number of even integers appearing among the d_i.

Theorem

$$Z_{\hat{G}}(h, e, f) \quad \longrightarrow \quad \prod_{s\ odd} O(M_s) \times \prod_{s\ even} Sp(M_s)$$

is an isomorphism of groups.

Exceptional Lie algebras

From now on, let $\mathfrak{g}$ be a Lie algebra of type G_2, F_4, E_6, E_7 or E_8.

Nilpotent orbits of such Lie algebras where characterized by $A.V.Alekseevskii$ (1978) and Sommers (1998). Moreover, R. Lawther and D. M. Testerman made some impressive hand computations of component groups (2011).

Exceptional Lie algebras

From now on, let $\mathfrak{g}$ be a Lie algebra of type G_2, F_4, E_6, E_7 or E_8.

Nilpotent orbits of such Lie algebras where characterized by $A.V.Alekseevskii$ (1978) and Sommers (1998). Moreover, R. Lawther and D. M. Testerman made some impressive hand computations of component groups (2011).
Summarizing, we have the following data:

Exceptional Lie algebras

From now on, let $\mathfrak{g}$ be a Lie algebra of type G_2, F_4, E_6, E_7 or E_8.

Nilpotent orbits of such Lie algebras where characterized by $A.V.Alekseevskii$ (1978) and Sommers (1998). Moreover, R. Lawther and D. M. Testerman made some impressive hand computations of component groups (2011).
Summarizing, we have the following data:

- List of all nilpotent orbits;

Exceptional Lie algebras

From now on, let $\mathfrak{g}$ be a Lie algebra of type G_2, F_4, E_6, E_7 or E_8.

Nilpotent orbits of such Lie algebras where characterized by *A.V.Alekseevskii* (1978) and Sommers (1998). Moreover, R. Lawther and D. M. Testerman made some impressive hand computations of component groups (2011).

Summarizing, we have the following data:

- List of all nilpotent orbits;
- Isomorphism types of the component groups of the orbits;

Exceptional Lie algebras

From now on, let $\mathfrak{g}$ be a Lie algebra of type G_2, F_4, E_6, E_7 or E_8.

Nilpotent orbits of such Lie algebras where characterized by $A.V.Alekseevskii$ (1978) and Sommers (1998). Moreover, R. Lawther and D. M. Testerman made some impressive hand computations of component groups (2011).
Summarizing, we have the following data:

- List of all nilpotent orbits;
- Isomorphism types of the component groups of the orbits;
- Explicit computations of generators of component groups.

Exceptional Lie algebras

From now on, let $\mathfrak{g}$ be a Lie algebra of type G_2, F_4, E_6, E_7 or E_8.

Nilpotent orbits of such Lie algebras where characterized by *A.V.Alekseevskii* (1978) and Sommers (1998). Moreover, R. Lawther and D. M. Testerman made some impressive hand computations of component groups (2011).
Summarizing, we have the following data:

- List of all nilpotent orbits;
- Isomorphism types of the component groups of the orbits;
- Explicit computations of generators of component groups.

Exceptional Lie algebras

From now on, let $\mathfrak{g}$ be a Lie algebra of type G_2, F_4, E_6, E_7 or E_8.

Nilpotent orbits of such Lie algebras where characterized by *A.V.Alekseevskii* (1978) and Sommers (1998). Moreover, R. Lawther and D. M. Testerman made some impressive hand computations of component groups (2011).
Summarizing, we have the following data:

- List of all nilpotent orbits;
- Isomorphism types of the component groups of the orbits;
- Explicit computations of generators of component groups.

Our goal is to overcome limits of the last point, providing a unified strategy and developing computational methods to find generators for component groups.

Definition

A Lie algebra $\mathfrak{g}$ is reductive if $\mathfrak{g} = \mathfrak{z}_{\mathfrak{g}}(\mathfrak{g}) \oplus [\mathfrak{g}, \mathfrak{g}]$.

Definition

A Lie algebra $\mathfrak{g}$ is reductive if $\mathfrak{g} = \mathfrak{z}_{\mathfrak{g}}(\mathfrak{g}) \oplus [\mathfrak{g}, \mathfrak{g}]$.

Fix an $\mathfrak{sl}_2$-triple (h, e, f) and let:

- $\mathfrak{c}_1 := \mathfrak{z}_{\mathfrak{g}}(h, e, f) := \{x \in \mathfrak{g} \mid [h, x] = [e, x] = [f, x] = 0\}$;
- $\hat{\mathfrak{c}}_2 := \mathfrak{z}_{\mathfrak{g}}(\mathfrak{c}_1) := \{x \in \mathfrak{g} \mid [y, x] = 0 \text{ for all } y \in \mathfrak{c}_1\}$;
- $\mathfrak{c}_2 := [\hat{\mathfrak{c}}_2, \hat{\mathfrak{c}}_2]$.

Definition

A Lie algebra $\mathfrak{g}$ is reductive if $\mathfrak{g} = \mathfrak{z}_{\mathfrak{g}}(\mathfrak{g}) \oplus [\mathfrak{g}, \mathfrak{g}]$.

Fix an $\mathfrak{sl}_2$-triple (h, e, f) and let:

- $\mathfrak{c}_1 := \mathfrak{z}_{\mathfrak{g}}(h, e, f) := \{x \in \mathfrak{g} \mid [h, x] = [e, x] = [f, x] = 0\}$;
- $\hat{\mathfrak{c}}_2 := \mathfrak{z}_{\mathfrak{g}}(\mathfrak{c}_1) := \{x \in \mathfrak{g} \mid [y, x] = 0 \text{ for all } y \in \mathfrak{c}_1\}$;
- $\mathfrak{c}_2 := [\hat{\mathfrak{c}}_2, \hat{\mathfrak{c}}_2]$.

Lemma

Any $\sigma \in Z_G(h, e, f)$ stabilizes $\mathfrak{c}_1, \mathfrak{c}_2, \hat{\mathfrak{c}}_2$.

Definition

A Lie algebra $\mathfrak{g}$ is reductive if $\mathfrak{g} = \mathfrak{z}_{\mathfrak{g}}(\mathfrak{g}) \oplus [\mathfrak{g}, \mathfrak{g}]$.

Fix an $\mathfrak{sl}_2$-triple (h, e, f) and let:

- $\mathfrak{c}_1 := \mathfrak{z}_{\mathfrak{g}}(h, e, f) := \{x \in \mathfrak{g} \mid [h, x] = [e, x] = [f, x] = 0\}$;
- $\hat{\mathfrak{c}}_2 := \mathfrak{z}_{\mathfrak{g}}(\mathfrak{c}_1) := \{x \in \mathfrak{g} \mid [y, x] = 0 \text{ for all } y \in \mathfrak{c}_1\}$;
- $\mathfrak{c}_2 := [\hat{\mathfrak{c}}_2, \hat{\mathfrak{c}}_2]$.

Lemma

Any $\sigma \in Z_G(h, e, f)$ stabilizes $\mathfrak{c}_1, \mathfrak{c}_2, \hat{\mathfrak{c}}_2$.

For nilpotent orbits of the exceptional Lie algebras there are three distinct cases:

1. $\mathfrak{c}_1$ is trivial;

Definition

A Lie algebra $\mathfrak{g}$ is reductive if $\mathfrak{g} = \mathfrak{z}_{\mathfrak{g}}(\mathfrak{g}) \oplus [\mathfrak{g}, \mathfrak{g}]$.

Fix an $\mathfrak{sl}_2$-triple (h, e, f) and let:

- $\mathfrak{c}_1 := \mathfrak{z}_{\mathfrak{g}}(h, e, f) := \{x \in \mathfrak{g} \mid [h, x] = [e, x] = [f, x] = 0\}$;
- $\hat{\mathfrak{c}}_2 := \mathfrak{z}_{\mathfrak{g}}(\mathfrak{c}_1) := \{x \in \mathfrak{g} \mid [y, x] = 0 \text{ for all } y \in \mathfrak{c}_1\}$;
- $\mathfrak{c}_2 := [\hat{\mathfrak{c}}_2, \hat{\mathfrak{c}}_2]$.

Lemma

Any $\sigma \in Z_G(h, e, f)$ stabilizes $\mathfrak{c}_1, \mathfrak{c}_2, \hat{\mathfrak{c}}_2$.

For nilpotent orbits of the exceptional Lie algebras there are three distinct cases:

1. $\mathfrak{c}_1$ is trivial;
2. $\mathfrak{c}_1$ has a trivial center, i.e. it is semisimple;

Definition

A Lie algebra $\mathfrak{g}$ is reductive if $\mathfrak{g} = \mathfrak{z}_{\mathfrak{g}}(\mathfrak{g}) \oplus [\mathfrak{g}, \mathfrak{g}]$.

Fix an $\mathfrak{sl}_2$-triple (h, e, f) and let:

- $\mathfrak{c}_1 := \mathfrak{z}_{\mathfrak{g}}(h, e, f) := \{x \in \mathfrak{g} \mid [h, x] = [e, x] = [f, x] = 0\}$;
- $\hat{\mathfrak{c}}_2 := \mathfrak{z}_{\mathfrak{g}}(\mathfrak{c}_1) := \{x \in \mathfrak{g} \mid [y, x] = 0 \text{ for all } y \in \mathfrak{c}_1\}$;
- $\mathfrak{c}_2 := [\hat{\mathfrak{c}}_2, \hat{\mathfrak{c}}_2]$.

Lemma

Any $\sigma \in Z_G(h, e, f)$ stabilizes $\mathfrak{c}_1, \mathfrak{c}_2, \hat{\mathfrak{c}}_2$.

For nilpotent orbits of the exceptional Lie algebras there are three distinct cases:

1. $\mathfrak{c}_1$ is trivial;
2. $\mathfrak{c}_1$ has a trivial center, i.e. it is semisimple;
3. $\mathfrak{c}_1$ has a non-trivial center.

$\mathfrak{c}_1$ is trivial

Observe that in this case $Z_G(h, e, f)$ is a finite group since its Lie algebra is zero.

Consider $Z_G(h) = \{g \in G \mid g(h) = h\}$. This group is connected and its Lie algebra is the reductive Lie algebra defined as follows:

$$\mathfrak{z}_{\mathfrak{g}}(h) = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Psi} \mathfrak{g}_\alpha$$

where $\Psi = \{\alpha \in \Phi \mid \alpha(h) = 0\}$. Fix a base $\Pi = \{\beta_1, \ldots, \beta_m\}$ and a canonical generating set $\{x_{\pm\beta_i}, h_{\beta_i}\}$ and let W_0 be the Weyl group of Ψ.

$\mathfrak{c}_1$ is trivial

Then

$$Z_G(h) = \bigsqcup_{w \in W_0} UHwU_w$$

where

$$U = \{e^{s_1 adx_{\beta_1}} \cdots e^{s_m adx_{\beta_m}} \text{ for } s_1, \ldots, s_m \in \mathbb{C}\}$$

$$H \text{ is a maximal torus of the form } \{h_1(t_1) \cdots h_l(t_l) \mid t_1, \ldots, t_l \in \mathbb{C}^*\}$$

$$h_i(t) = w_{\beta_i}(t) w_{\beta_i}(1)^{-1}, w_{\beta_i}(t) = e^{tadx_{\beta_i}} e^{-t^{-1}adx_{-\beta_i}} e^{tadx_{\beta_i}}$$

$$U_w = \{e^{u_{i_1} adx_{\beta_{i_1}}} \cdots e^{u_{i_m} adx_{\beta_{i_m}}} \text{ for } u_1, \ldots, u_m \in \mathbb{C}\}$$

$$\Psi_w = \{\beta_{i_1}, \ldots, \beta_{i_n}\} \subset \Phi \text{ the set of all } \beta_i \text{ such that } w(\beta_i) \text{ is a negative root .}$$

$\mathfrak{c}_1$ is trivial

Then

$$Z_G(h) = \bigsqcup_{w \in W_0} UHwU_w$$

where

$$U = \{e^{s_1 adx_{\beta_1}} \cdots e^{s_m adx_{\beta_m}} \text{ for } s_1, \dots, s_m \in \mathbb{C}\}$$

$$H \text{ is a maximal torus of the form } \{h_1(t_1) \cdots h_l(t_l) \mid t_1, \dots, t_l \in \mathbb{C}^*\}$$

$$h_i(t) = w_{\beta_i}(t) w_{\beta_i}(1)^{-1}, w_{\beta_i}(t) = e^{tadx_{\beta_i}} e^{-t^{-1} adx_{-\beta_i}} e^{tadx_{\beta_i}}$$

$$U_w = \{e^{u_{i_1} adx_{\beta_{i_1}}} \cdots e^{u_{i_m} adx_{\beta_{i_m}}} \text{ for } u_1, \dots, u_m \in \mathbb{C}\}$$

$$\Psi_w = \{\beta_{i_1}, \dots, \beta_{i_n}\} \subset \Phi \text{ the set of all } \beta_i \text{ such that } w(\beta_i) \text{ is a negative root .}$$

At this point it suffices to set $g(e) = e$ for $g \in Z_G(h)$ and solve with respect to the indeterminates $s_1, \dots, s_m, t_1, \dots, t_l, u_1, \dots, u_m$.

$\mathfrak{c}_1$ is semisimple

Set $\mathfrak{c} := \mathfrak{c}_1 \oplus \mathfrak{c}_2$ and $\mathfrak{c}^\perp = \{x \in \mathfrak{g} \mid k(x, y) = 0 \text{ for all } y \in \mathfrak{c}\}$.

The Killing form is non-degenerate, being $\mathfrak{g}$ semisimple. This implies that $\mathfrak{c} \cap \mathfrak{c}^\perp = 0$, hence $\mathfrak{g} = \mathfrak{c} \oplus \mathfrak{c}^\perp$.

$\mathfrak{c}_1$ is semisimple

Set $\mathfrak{c} := \mathfrak{c}_1 \oplus \mathfrak{c}_2$ and $\mathfrak{c}^\perp = \{x \in \mathfrak{g} \mid k(x, y) = 0 \text{ for all } y \in \mathfrak{c}\}$.

The Killing form is non-degenerate, being $\mathfrak{g}$ semisimple. This implies that $\mathfrak{c} \cap \mathfrak{c}^\perp = 0$, hence $\mathfrak{g} = \mathfrak{c} \oplus \mathfrak{c}^\perp$.

Lemma

$[\mathfrak{c}, \mathfrak{c}^\perp] \subset \mathfrak{c}^\perp$, *i.e.* $\mathfrak{c}^\perp$ *is a* $\mathfrak{c}$*-module.*

$\mathfrak{c}_1$ is semisimple

Let $\sigma \in Z_G(h, e, f)$.

$\mathfrak{c}_1$ is semisimple

Let $\sigma \in Z_G(h, e, f)$.

- $\sigma|_{\mathfrak{c}_1} \in Aut(\mathfrak{c}_1) = Aut(\mathfrak{c}_1)^0 \rtimes \Gamma_{\mathfrak{c}_1} = Z_G(h, e, f)^0 \rtimes \Gamma_{\mathfrak{c}_1}$. In other words, we can find $\phi \in Z_G(h, e, f)^0$ such that $\sigma\phi|_{\mathfrak{c}_1} \in \Gamma_{\mathfrak{c}_1}$.

$\mathfrak{c}_1$ is semisimple

Let $\sigma \in Z_G(h, e, f)$.

- $\sigma|_{\mathfrak{c}_1} \in Aut(\mathfrak{c}_1) = Aut(\mathfrak{c}_1)^0 \rtimes \Gamma_{\mathfrak{c}_1} = Z_G(h, e, f)^0 \rtimes \Gamma_{\mathfrak{c}_1}$. In other words, we can find $\phi \in Z_G(h, e, f)^0$ such that $\sigma\phi|_{\mathfrak{c}_1} \in \Gamma_{\mathfrak{c}_1}$.
- $\sigma|_{\mathfrak{c}_2} \in Z_{Aut(\mathfrak{c}_2)}(h, e, f)$ but its Lie algebra is $\mathfrak{z}_{\mathfrak{c}_2} \subset \mathfrak{z}_{\mathfrak{g}} \cap \mathfrak{c}_2 = 0$. Hence $Z_{Aut(\mathfrak{c}_2)}(h, e, f)$ is finite.

$\mathfrak{c}_1$ is semisimple

Let $\sigma \in Z_G(h, e, f)$.

- $\sigma|_{\mathfrak{c}_1} \in Aut(\mathfrak{c}_1) = Aut(\mathfrak{c}_1)^0 \rtimes \Gamma_{\mathfrak{c}_1} = Z_G(h, e, f)^0 \rtimes \Gamma_{\mathfrak{c}_1}$. In other words, we can find $\phi \in Z_G(h, e, f)^0$ such that $\sigma\phi|_{\mathfrak{c}_1} \in \Gamma_{\mathfrak{c}_1}$.
- $\sigma|_{\mathfrak{c}_2} \in Z_{Aut(\mathfrak{c}_2)}(h, e, f)$ but its Lie algebra is $\mathfrak{z}_{\mathfrak{c}_2} \subset \mathfrak{z}_{\mathfrak{g}} \cap \mathfrak{c}_2 = 0$. Hence $Z_{Aut(\mathfrak{c}_2)}(h, e, f)$ is finite.

$\mathfrak{c}_1$ is semisimple

Let $\sigma \in Z_G(h, e, f)$.

- $\sigma|_{\mathfrak{c}_1} \in Aut(\mathfrak{c}_1) = Aut(\mathfrak{c}_1)^0 \rtimes \Gamma_{\mathfrak{c}_1} = Z_G(h, e, f)^0 \rtimes \Gamma_{\mathfrak{c}_1}$. In other words, we can find $\phi \in Z_G(h, e, f)^0$ such that $\sigma\phi|_{\mathfrak{c}_1} \in \Gamma_{\mathfrak{c}_1}$.
- $\sigma|_{\mathfrak{c}_2} \in Z_{Aut(\mathfrak{c}_2)}(h, e, f)$ but its Lie algebra is $\mathfrak{z}_{\mathfrak{c}_2} \subset \mathfrak{z}_{\mathfrak{g}} \cap \mathfrak{c}_2 = 0$. Hence $Z_{Aut(\mathfrak{c}_2)}(h, e, f)$ is finite.

All this implies that we can determine a finite set U of automorphisms of $\mathfrak{c}$ that stabilize $\mathfrak{c}_1$ and $\mathfrak{c}_2$ and such that extending these automorphisms to all of $\mathfrak{g}$ gives at least an element for each component of $Z_G(h, e, f)$.

It turns out that an arbitrary element τ of U can be extended in a finite number of ways.

$\mathfrak{c}_1$ is semisimple

Let $V = \mathfrak{c}^\perp$ and write $V = V_1 \oplus \cdots \oplus V_m$, where V_i are the irreducible $\mathfrak{c}$-modules. V can be assumed to be multiplicity free.

$\mathfrak{c}_1$ is semisimple

Let $V = \mathfrak{c}^{\perp}$ and write $V = V_1 \oplus \cdots \oplus V_m$, where V_i are the irreducible $\mathfrak{c}$-modules. V can be assumed to be multiplicity free.

- Fix a canonical generating set $\{h_i, x_i, y_i \text{ for } 1 \leq i \leq s\}$ and compute the maximal vectors $\{v_1, \ldots, v_m\}$, i.e. vectors such that $x_j \cdot v_i = 0$ for all j.

$\mathfrak{c}_1$ is semisimple

Let $V = \mathfrak{c}^\perp$ and write $V = V_1 \oplus \cdots \oplus V_m$, where V_i are the irreducible $\mathfrak{c}$-modules. V can be assumed to be multiplicity free.

- Fix a canonical generating set $\{h_i, x_i, y_i \text{ for } 1 \leq i \leq s\}$ and compute the maximal vectors $\{v_1, \ldots, v_m\}$, i.e. vectors such that $x_j \cdot v_i = 0$ for all j.
- Consider the canonical generating set $\{\tau(h_i), \tau(x_i), \tau(y_i) \text{ for } 1 \leq i \leq s\}$ and compute the maximal vectors $\{w_1, \ldots, w_m\}$.

$\mathfrak{c}_1$ is semisimple

Let $V = \mathfrak{c}^{\perp}$ and write $V = V_1 \oplus \cdots \oplus V_m$, where V_i are the irreducible $\mathfrak{c}$-modules. V can be assumed to be multiplicity free.

- Fix a canonical generating set $\{h_i, x_i, y_i \text{ for } 1 \leq i \leq s\}$ and compute the maximal vectors $\{v_1, \ldots, v_m\}$, i.e. vectors such that $x_j \cdot v_i = 0$ for all j.
- Consider the canonical generating set $\{\tau(h_i), \tau(x_i), \tau(y_i) \text{ for } 1 \leq i \leq s\}$ and compute the maximal vectors $\{w_1, \ldots, w_m\}$.
- Fix i_0 with $1 \leq i_0 \leq m$ and define integers $\nu_1^{i_0}, \ldots, \nu_s^{i_0}$ by $h_i \cdot v_{i_0} = \nu_i^{i_0} v_{i_0}$.

$\mathfrak{c}_1$ is semisimple

Let $V = \mathfrak{c}^{\perp}$ and write $V = V_1 \oplus \cdots \oplus V_m$, where V_i are the irreducible $\mathfrak{c}$-modules. V can be assumed to be multiplicity free.

- Fix a canonical generating set $\{h_i, x_i, y_i \text{ for } 1 \leq i \leq s\}$ and compute the maximal vectors $\{v_1, \ldots, v_m\}$, i.e. vectors such that $x_j \cdot v_i = 0$ for all j.
- Consider the canonical generating set $\{\tau(h_i), \tau(x_i), \tau(y_i) \text{ for } 1 \leq i \leq s\}$ and compute the maximal vectors $\{w_1, \ldots, w_m\}$.
- Fix i_0 with $1 \leq i_0 \leq m$ and define integers $\nu_1^{i_0}, \ldots, \nu_s^{i_0}$ by $h_i \cdot v_{i_0} = \nu_i^{i_0} v_{i_0}$.
- If there is a j_0 such that $\tau(h_i) \cdot w_{j_0} = \nu_i^{i_0} w_{j_0}$ for $1 \leq i \leq s$ then the extension of τ has to map $v_{i_0} \to k_{i_0,j_0} w_{j_0}$. We get then a system of equations between generators of $\mathfrak{g}$ in the indeterminates k_{i_0,j_0}.

$\mathfrak{c}_1$ has non-trivial center

In this case we do not know what the possible restrictions of an element in $Z_G(h, e, f)$ to $\mathfrak{c}_1$ can be. We proceed as follows:

$\mathfrak{c}_1$ has non-trivial center

In this case we do not know what the possible restrictions of an element in $Z_G(h, e, f)$ to $\mathfrak{c}_1$ can be. We proceed as follows:

- Compute $\mathfrak{c}_1$ and then $\mathfrak{c}_2$;

$\mathfrak{c}_1$ has non-trivial center

In this case we do not know what the possible restrictions of an element in $Z_G(h, e, f)$ to $\mathfrak{c}_1$ can be. We proceed as follows:

- Compute $\mathfrak{c}_1$ and then $\mathfrak{c}_2$;
- Compute the finite group of diagram automorphisms $Z_{Aut(\mathfrak{c}_2)}(h, e, f)$ as before;

$\mathfrak{c}_1$ has non-trivial center

In this case we do not know what the possible restrictions of an element in $Z_G(h, e, f)$ to $\mathfrak{c}_1$ can be. We proceed as follows:

- Compute $\mathfrak{c}_1$ and then $\mathfrak{c}_2$;
- Compute the finite group of diagram automorphisms $Z_{Aut(\mathfrak{c}_2)}(h, e, f)$ as before;
- Try to extend these morphisms directly from $\mathfrak{c}_2$ to $\mathfrak{g}$;

$\mathfrak{c}_1$ has non-trivial center

In this case we do not know what the possible restrictions of an element in $Z_G(h, e, f)$ to $\mathfrak{c}_1$ can be. We proceed as follows:

- Compute $\mathfrak{c}_1$ and then $\mathfrak{c}_2$;
- Compute the finite group of diagram automorphisms $Z_{Aut(\mathfrak{c}_2)}(h, e, f)$ as before;
- Try to extend these morphisms directly from $\mathfrak{c}_2$ to $\mathfrak{g}$;
- We need to use different strategies case by case.