THREE SURPRISES ABOUT PERMUTATION REPRESENTATIONS

Martin Gallauer

3 June 2024

1. Groups and their actions:
2. algebraic,
3. geometric and
4. combinatorial aspects

PARADIGM

$$G \quad \circlearrowleft \quad \text{objects}$$

- study objects with action, ideally classify them
- draw conclusions about G

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$$G \circlearrowleft \text{sets}$$

- $S = \coprod G/H_i, \quad G/H \cong G/H' \iff (H) = (H')$
- learn about subgroups up to conjugacy

Algebra

kG-MODULES

$G \circlearrowleft k$-vector spaces

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DEFINITION

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CHARACTERISTIC 0

- semisimple
- character theory and applications

$G \quad \circlearrowleft \quad k$-vector spaces

DEFINITION

permutation module $= k(S)$

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CHARACTERISTIC p

- wild classification problem
- ?

FIRST SUPRISE

NOTE

S indecomposable $\not\Rightarrow$ $k(S)$ indecomposable

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COROLLARY

$$\mathrm{D}^b(kG) = \langle p\text{-permutation modules}\rangle^{\triangle}$$

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REMARK

In fact, for any R, the right-hand side is characterized by the cohomological singularity.

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Permutation generation holds for R regular.

QUESTION (MATHEW)

When does permutation generation hold for spectral coefficients?

Geometry

WEAKER CLASSIFICATION PROBLEM

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DICTIONARY

T	$\triangle, \natural$	$\otimes$	0	$\mathbb{1}$
R	$+$	$\cdot$	0	1

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SPECTRUM (BALMER)

$\mathrm{Spc}(T)$ space encoding tt-ideals in T

BACK TO kG-MODULES

Back to kG-modules

Theorem (Benson-Carlson-Rickard)

$$\mathrm{Spc}(\mathrm{D}^{\mathrm{b}}(kG)) = \mathrm{Spec}^{\mathrm{h}}(\mathrm{H}^*(G; k))$$

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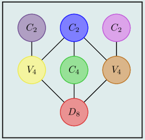
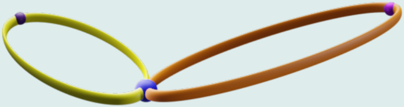
$G = D_8$





DERIVING PERMUTATION MODULES

$$\begin{array}{ccc} G\text{-sets} & \longrightarrow & kG\text{-mod} \\ & & \big\downarrow \\ & & \mathrm{D^b}(kG) \end{array}$$

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$\mathcal{K}(G) =$

- perfect complexes of cohomological Mackey functors (Thévenaz-Webb)
- geometric Artin motives (Voevodsky)
- compact spectral modules over constant Mackey functor (…, Fuhrmann)

TRANSLATION

$$\mathrm{Spc}(\mathcal{K}(G)) \quad \hookleftarrow\!\!\!\!\!\!\!\circ\;\;\; \quad V_G$$

THEOREM (BALMER-G.)

Complete description of $\mathrm{Spc}(\mathcal{K}(G))$. In particular,

$$\mathrm{Spc}(\mathcal{K}(G)) = \amalg_{(H)_p} V_{G//H}$$

SECOND SURPRISE

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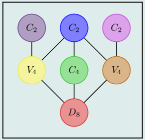

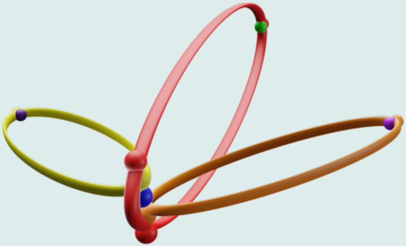

Combinatorics

DEFINITION

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$G = C_5$

$$k \to m_1$$
$$k \to kC_5 \to k \oplus kC_5 \to m_2$$
$$k \to kC_5 \to k \oplus kC_5 \to kC_5 \to m_3$$
$$k \to kC_5 \to m_4$$
$$kC_5 \to m_5$$

THEOREM IN PROGRESS (G.-WALSH)

Let $G = C_n$ with p-Sylow C_{p^r}.

1. $\mathrm{ppdim}_k(C_n) = \mathrm{ppdim}_k(C_{p^r})$.
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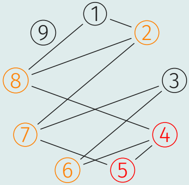

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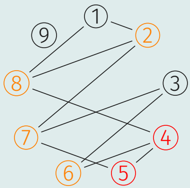

$$m_1 \to m_9 \to m_9 \oplus m_3 \to m_4$$
$$m_3 \to m_9 \to m_9 \oplus m_1 \to m_4$$

$$\mathrm{ppdim}_k(C_9) = 2$$

COROLLARY

$\mathrm{ppdim}_k(C_p) = p - 2$

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$\mathrm{ppdim}_k(C_{p^r})$

r \ p	2	3	5	7	11	13	17	⋯	31
1	0	1	3	5	9	11	15		29
2	1	2	5	7	13	15	21		39
3	1	3	7	11	19	23	31		59
4	2	4	9	13	23	28	37		71
5	2	5	11	17	29	35	47		89
6	2	6	13	19	33	40	?		?
7	3	7	15	23	39	47	?		?
8	3	8	17	25	?	?	?		?

Thank you!