The lower central series of the unit group of an integral group ring

Sugandha Maheshwary

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June 3, 2024

Notation

- G: a group
- $\mathbb{Z}G$: integral group ring of G, $\{\sum_{g\in G}\alpha_g g : \alpha_g \in \mathbb{Z}, g \in G\}$
- $\mathcal{U} := \mathcal{U}(\mathbb{Z}G)$: unit group of $\mathbb{Z}G$
- $\epsilon : \mathbb{Z}G \to \mathbb{Z}$: augmentation homomorphism ($g \to 1$).
- $\epsilon(u) = \pm 1,\ u \in \mathcal{U}$.
- $\mathcal{V} := \mathcal{V}(\mathbb{Z}G)$: subgroup formed by elements of $\mathcal{U}$ of augmentation 1, the subgroup of normalized units in $\mathcal{U}$.
- $\mathcal{U} = \pm\mathcal{V}$

$$\gamma_1(\mathcal{V}) = \mathcal{V},\ \gamma_2(\mathcal{V}) = \mathcal{V}',\ \gamma_i(\mathcal{V}) = [\gamma_{i-1}(\mathcal{V}), \mathcal{V}],\ i \geq 2.$$

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Classify the groups G for which $\mathcal{V}(\mathbb{Z}G)' = G'$.

- The lower central series of G and $\mathcal{V}(\mathbb{Z}G)$ coincide $\iff \mathcal{V}(\mathbb{Z}G) = G$.
- If G is finite, $\mathcal{V}(\mathbb{Z}G) = G \iff G$ is an abelian group of exponent 2, 3, 4 or 6, or $G = Q_8 \times E$, where E denotes an elementary abelian 2-group and Q_8 is the quaternion group of order 8.

Theorem

For a finite group G, $\mathcal{V}(\mathbb{Z}G)' = G'$, if and only if, G is an abelian group or a Hamiltonion 2-group.

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Let G be a finite group G, then exactly one of the following occurs:

- *G is abelian (and hence so is $\mathcal{V}(\mathbb{Z}G)$).*
- *G is a Hamiltonian-2 group and $\mathcal{V}(\mathbb{Z}G) = \{\pm g \mid g \in G\}$.*
- *$\mathcal{V}(\mathbb{Z}G)$ contains a free subgroup of rank 2.*

The problem remains open for an arbitrary group.
This problem is motivated by an analogous question about the upper central series of $\mathcal{V}(\mathbb{Z}G)$.

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$$\langle 1 \rangle = \mathcal{Z}_0(\mathcal{V}) \subseteq \mathcal{Z}_1(\mathcal{V}) \subseteq ... \mathcal{Z}_n(\mathcal{V}) \subseteq \mathcal{Z}_{n+1}(\mathcal{V}) \subseteq ...$$

[AHP93, AP93] Let G be a finite group.

- the central height of $\mathcal{V}$, i.e., the smallest integer $n \geq 0$ such that $\mathcal{Z}_n(\mathcal{V}) = \mathcal{Z}_{n+1}(\mathcal{V})$, is at most 2.
- the central height of $\mathcal{V}$ is 2 if, and only if, G is a Q^* group, i.e., if it has an element a of order 4 and an abelian subgroup H of index 2, which is not an elementary abelian 2-group, such that $G = \langle H, a \rangle, h^a = h^{-1}, \forall\, h \in H$ and $a^2 = b^2$, for some $b \in H$.
- In case the central height of $\mathcal{V}$ is 2, then $\mathcal{Z}_2(\mathcal{V}) = T\mathcal{Z}_1(\mathcal{V})$, where $T = \langle b \rangle \oplus E_2$, E_2 being an elementary abelian 2- group.

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Cut-groups

- If a group G is not a $Q*$ group, the central height of $\mathcal{V}$ must be 0 or 1.
- Central height 0 essentially means $\mathcal{Z}(\mathcal{V}(\mathbb{Z}G)) = 1$.
- Since $\mathcal{Z}(G) \subseteq \mathcal{Z}(\mathcal{V}(\mathbb{Z}G))$, the group G must have trivial centre and $\mathcal{Z}(G) = \mathcal{Z}(\mathcal{V}(\mathbb{Z}G))$.

Definition[BMP17]

In case $\mathcal{Z}(\mathcal{V}(\mathbb{Z}G)) = \mathcal{Z}(G)$ i.e., all **c**entral **u**nits are **t**rivial, G is called a **cut**-group, or a group with the cut-property.

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Problem 2

Given a group G, when does lower central series of $\mathcal{V}(\mathbb{Z}G)$ stabilize?

- No bound is known for the number of terms in the lower central series of $\mathcal{V}(\mathbb{Z}G)$.
- If $\mathcal{V}(\mathbb{Z}G)$ is nilpotent, the number of terms in both the upper and the lower central series coincide.
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The termination of the lower central series of $\mathcal{V}(\mathbb{Z}G)$

Theorem [SZ77]

$\mathcal{V}(\mathbb{Z}G)$ is nilpotent, if and only if, G is nilpotent and the torsion subgroup T of G satisfies one of the following conditions:

(i) T is central in G.

(ii) T is an abelian 2-group and for $x \in G$, $t \in T$, $xtx^{-1} = t^{\pm 1}$.

(iii) $T = E \times Q_8$, where E is an elementary abelian 2-group and Q_8 is the quaternion group of order 8. Moreover, E is central in G and conjugation by $x \in G$, induces on Q_8, one of the four inner automorphisms.

If $\mathcal{V}(\mathbb{Z}G)$ is not nilpotent, apparently, there is no answer for the stated problem.

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$\mathcal{V}(\mathbb{Z}G)$ is nilpotent, if and only if, G is nilpotent and the torsion subgroup T of G satisfies one of the following conditions:

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S. Maheshwary, [Mah21]

Problem 3

Given a group G, when is $\mathcal{V}(\mathbb{Z}G)$ residually nilpotent?

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A group G is said to be *residually nilpotent*, if the *nilpotent residue* defined by

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For a finite group G, the group $\mathcal{V}(\mathbb{Z}G)$ is residually nilpotent, if and only if, G is a nilpotent group which is a p-abelian group, i.e., the commutator subgroup G' is a p-group, for some prime p.

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Units and augmentation powers in integral group rings

Maheshwary, S. and Passi, I. B. S., *J. Group Theory*,[MP20]

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- *(i)* $T = \{1\}$;
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In particular, if G is a nilpotent group with its torsion subgroup $\{2, 3\}$-torsion-free, then, $\mathcal{V}(\mathbb{Z}G)$ has trivial Δ-adic residue only if either G is a torsion-free group or its torsion subgroup is a p-group which has no element of infinite p-height.

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Let G be an abelian group and let T be its torsion subgroup. Then, $\mathcal{V}_\omega(\mathbb{Z}G) = \{1\}$ if, and only if, $\mathcal{V}_\omega(\mathbb{Z}T) = \{1\}$.

- Examined the class $\mathcal{C}$ of groups G with $\mathcal{V}_\omega(\mathbb{Z}G) = \{1\}$, and prove that a group G belongs to $\mathcal{C}$ if all its quotients $G/\gamma_n(G)$ do so.
- Also, examined the groups G which have the property that the dimension series $\{D_{n,\mathbb{Q}}(G)\}_{n\geq 1}$ over the rationals has non-trivial intersection while $\{D_n(G)\}_{n\geq 1}$, the one over the integers, has trivial intersection.

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Given a group G, describe $\gamma_i(\mathcal{V}(\mathbb{Z}G))/\gamma_{i+1}(\mathcal{V}(\mathbb{Z}G))$, for $i \geq 0$.

Theorem ([SGV97])

If $G = S_3$, the symmetric group on 3 elements, then

- $\mathcal{V}/\mathcal{V}'$ *is isomorphic to the Klein's 4 group,*
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Theorem ([SG01])

If $G = D_8$, the dihedral group on 4 elements, then

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$[H : Z(H)] < \infty \implies |H'| < \infty.$

B.H.Neumann

If H is finitely generated, then $|H'| < \infty \implies [H : Z(H)] < \infty$

(1): Does $[H : H'] < \infty$ imply $|Z(H)| < \infty$?

N: the direct product of countably many Prüfer 2-groups C_{2^∞}, x be an involution acting on each of these direct factors by inversion. Then $G = N \rtimes \langle x \rangle$ has infinite center, consisting of 1 and all the involutions in N, but has finite abelianization, as $G' = N$.

(2): Does $|Z(H)| < \infty$ imply $[H : H'] < \infty$?

$SL_2(\mathbb{Z}[\sqrt{-2}])$ has infinite abelianization, but finite centre. In fact, Any non-abelian free group (centre trivial), rank of abelianization-same as number of generators.

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(1): Does $[H : H'] < \infty$ imply $|Z(H)| < \infty$?

N: the direct product of countably many Prüfer 2-groups C_{2^∞}, x be an involution acting on each of these direct factors by inversion. Then $G = N \rtimes \langle x \rangle$ has infinite center, consisting of 1 and all the involutions in N, but has finite abelianization, as $G' = N$.

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Question: What if $H = \mathcal{V}(\mathbb{Z}G)$?

Bachle et al., [BJJ+23], Abelianization and fixed point properties of units in integral group rings

If $\mathcal{O}$ is an order in a finite-dimensional semi-simple rational algebra with unit group $U = \mathrm{U}(\mathcal{O})$, then

$$\operatorname{rank} U/U' \geqslant \operatorname{rank} K_1(\mathcal{O}) = \operatorname{rank} \mathcal{Z}(U),$$

where $K_1(\mathcal{O}) = \mathrm{GL}(\mathcal{O})/\,\mathrm{GL}(\mathcal{O})'$, and $\operatorname{rank} A$ denotes the torsion-free rank of a finitely generated abelian group A

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(R2) Assume $\mathcal{Z}(\mathcal{V})$ is finite. Is $\mathcal{V}/\mathcal{V}'$ also finite?

(E1) Is $\exp \mathcal{V}/\mathcal{V}' = \exp G/G'$?

(E2) Does $\exp \mathcal{V}/\mathcal{V}'$ divide $\exp G$?

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Theorem

Let G be a finite group and let $\mathcal{B}$ be the subgroup of $V = \mathcal{V}(\mathbb{Z}G)$, generated by the elements of G, the bicyclic and the Bass units of $\mathbb{Z}G$. If $\mathcal{B}$ has finite index in V, then $\operatorname{rank} V/V' = \operatorname{rank} \mathcal{Z}(V)$, i.e., (R1) has a positive answer.

Corollary

Let G be a dihedral group and let $V = \mathrm{V}(\mathbb{Z}G)$. Then $\operatorname{rank} \mathcal{Z}(V) = \operatorname{rank} V/V'$, i.e., (R1) has a positive answer.

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Abelianization of the unit group of an integral group ring

A. Bachle, S. Maheshwary and L. Margolis, [BMM21]

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Bicyclic Units

For a subgroup H of G and an element g in G, $\widetilde{H} = \sum_{h \in H} h \in \mathbb{Z}G$ and $\widetilde{g} = \widetilde{\langle g \rangle}$. For $g, h \in G$

$$b(g,h) := 1 + (1-h)g\widetilde{h},$$

denotes a *bicyclic unit* in $\mathrm{V}(\mathbb{Z}G)$.

Bicyclic Units

Let $g, h \in G$ be such that h is of order n. Then

$$\prod_{k=1}^{n} [b(g,h)^{-1}, h^k] = b(g,h)^n.$$

In particular, $\varphi(b(g,h))^n = 1$.

Bass units

If $g \in G$ is of order n and k, m are positive integers such that k is coprime to n and $k^m \equiv 1 \mod n$, then

$$u_{k,m}(g) := (1 + g + g^2 + ... + g^{k-1})^m + \frac{1 - k^m}{n}\widetilde{g}$$

is a *Bass unit*.

Bass units

Let $g \in G$ be an element of order n and let l, m be integers such that $l^m \equiv 1 \mod n$. Assume that $g \sim_G g^l$, say $g^h = g^l$ for some $h \in G$, and let s be the order of l in $\mathrm{U}(\mathbb{Z}/n\mathbb{Z})$. Then

$$\prod_{i=1}^{s-1}[u_{l,m}(g)^{-1}, h^i] = u_{l,m}(g)^s.$$

In particular, $\varphi(u_{l,m}(g))^s = 1$.

Theorem

Proposition Let G be a dihedral group of order $2p$*, where p is an odd prime, and let* $V = \mathcal{V}(\mathbb{Z}G)$*. Then* $\exp V/V' = \exp G/G'$*, i.e., (E1) holds for G.*

Theorem

Let G be a group and let $V = \mathcal{V}(\mathbb{Z}G)$*.*

1. *If G is of order at most* 15*, then (R1) and (E1) have positive answers for G.*
2. *There are non-abelian groups of order* 16 *for which (R1) has a positive answer. There is a group of order* 16 *for which (R2), and hence also (R1), has a negative answer.*

Description of V/V', for groups of order ≤ 16, [BMM21]

- If G is an abelian cut-group, i.e., of exponent 2,3,4 or 6, then $V = G$, and $V/V' = V = G$.
- If G is an abelian group (of any exponent), then $V/V' = V = G \times F$, where F is f.g. free group of rank $\frac{1}{2}(|G| + n_2 - 2c + 1)$, where $|G|$ denotes the order of the group G, n_2 is the number of elements of order 2 in G and c is the number of cyclic subgroups of G.
- Computations for non-abelian groups.

$|G|$

6 $G \simeq S_3$,
$V/V' \simeq C_2 \times C_2$.
(R1) ✓ (E1) ✓

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- $G \simeq Q_8$, then $V = G$. Hence, $V/V' = G/G' = C_2 \times C_2$.
- $G \simeq D_8$, then $V/V' = C_2^4$.

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Description of V/V', for groups of order 16, [BMM21]

16

- $G = P := \langle a, b \mid a^4 = 1, b^4 = 1, a^b = a^{-1} \rangle$
  $V/V' \simeq C_4 \times C_2^7$.
  (R1) ✓ (E1) ✓
- If $G = D_{16}^{+} := \langle a, b \mid a^8 = b^2 = 1, a^b = a^5 \rangle$;
  $V/V' \cong C_\infty \times C_4 \times C_2^5$. (R1) × (E1) ✓
- $G = D_{16}$, the dihedral group of order 16
  (R1) ✓ (E1) ?
- If $G = D := \langle a, b, c \mid a^2 = b^2 = c^4 = 1, a^c = a, b^c = b, a^b = c^2 ab \rangle$ or $G = D_{16}^{-} := \langle a, b \mid a^8 = b^2 = 1, a^b = a^3 \rangle$ the unit group $\mathrm{V}(\mathbb{Z}G)$ has also been studied in and one could, in principle, compute the abelianization of their unit groups, analogous to the case of D_{16}^{+}.
- If $G = H := \langle a, b \mid a^4 = b^4 = (ab)^2 = 1, (a^2)^b = a^2 \rangle$
  we cannot conclude if the abelianization of the unit group for this group is finite or not.
- For $G = \langle a, b \mid a^8 = 1, b^2 = a^4, a^b = a^{-1} \rangle$
  V has infinite abelianization, as G it is not a cut group.
  (R1) ? (E1) ?

A. Bachle, S. Maheshwary and L. Margolis, [BMM21]

Theorem

Let G be a finite group such that $V = \mathcal{V}(\mathbb{Z}G)$ has a free normal complement, i.e., $V = F \rtimes G$ for some infinite cyclic or non-abelian free group F. Then $\operatorname{rank} V/V' = \operatorname{rank} \mathcal{Z}(V) = 0$ and $\exp V/V' = \exp G/G'$, i.e., (R1) and (E1) have positive answers in this case.

- $G = S_3$: $\exp G/G' = 2$, $V/V' \cong C_2 \times C_2$.
- $G = D_8$: $\exp G/G' = 2$, $V/V' \cong C_2^4$.
- $G = T$: $\exp G/G' = 4$, $V/V' \cong C_4 \times C_2$.
- $G = P$: $\exp G/G' = 4$, $V/V' \cong C_4 \times C_2^7$.

The lower central series of $\mathcal{V}$

S. Maheshwary, [Mah21]

Problem 5

For a group G, give a description of the terms in the lower central series of $\mathcal{V}(\mathbb{Z}G)$.

The answer is of course trivial if G is an abelian group.

Theorem ([SG00])

If $G = A_4$, the alternating group on 4 elements, then

- $\mathcal{V}/\mathcal{V}'$ *is isomorphic to the cyclic group of order 3 and*
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For a group G, give a description of the terms in the lower central series of $\mathcal{V}(\mathbb{Z}G)$.

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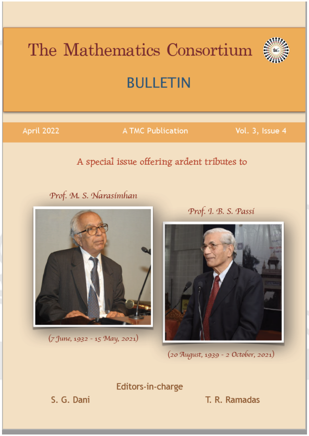
Dedicated to Late Prof. I. B. S. Passi

S.Maheshwary, *The Life and works of Profesor I.B.S. Passi*, [Mah22]



The Mathematics Consortium

BULLETIN

April 2022 A TMC Publication Vol. 3, Issue 4

A special issue offering ardent tributes to

Prof. M. S. Narasimhan

(7 June, 1932 - 15 May, 2021)

Prof. I. B. S. Passi

(20 August, 1939 - 2 October, 2021)

Editors-in-charge

S. G. Dani T. R. Ramadas

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THANK YOU!!!