The Module Structure of a Group Action on a Ring

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Context

Notation: k a field, of finite characteristic p to be interesting, occasionally assumed to be algebraically closed to avoid trivialities.

S a noetherian k-algebra graded by $\mathbb{N}$, e.g. a polynomial ring $S = k[x_1, \ldots, x_n]$.

G a finite group, which acts on S preserving the grading.

Any module over S or some noetherian graded sub-algebra will be assumed to be graded and finite, meaning finitely generated, not finite cardinality.

This is the context of classical invariant theory, but we are interested in the structure of S as a kG-module, not just the invariants.

Motivation

This is a natural problem in representation theory.

For the Lyndon-Hochschild-Serre spectral sequence arising from groups $H \lhd G$,

$$H^*(G/H; H^*(H; k)) \Longrightarrow H^*(G; k),$$

we need to understand the action of G/H on $H^*(H; k)$.

In chromatic homotopy theory we have the Adams-Novikov spectral sequence

$$H^*(\mathbb{G}_n; E_n) \Longrightarrow \pi_* L_{K(n)} S^0.$$

Maybe restrict to a finite subgroup.

We would also like some information about the multiplication.

Motivation

Galois Module Theory: L a number field, G acts, $K = L^G$.

Regard $\mathcal{O}_L$ as an $\mathcal{O}_K G$-module, or perhaps just as a $\mathbb{Z}G$-module.

$\mathcal{O}_L$ has a free submodule of finite index. It is locally free if L/K is tamely ramified.

From yesterday:

G acts on a curve C, hence on $H^0(C, \Omega_{C/k}^{\otimes m})$.

Automorphisms of $H^*(BT^n; \mathbb{F}_p)$.

Example

Field k of characteristic 3, $U = U_3(\mathbb{F}_3) = \left\{ \begin{pmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix} : * \in \mathbb{F}_3 \right\}$

Natural module V^*, basis x^*, y^*, z^*.

Ring $S = S(V^*) = k[V]$

Invariants:

$$\begin{aligned}
d_x &= x^* && \text{degree 1} \\
d_y &= \prod_{\lambda \in \mathbb{F}_3} (y^* + \lambda x^*) = y^{*3} - y^* x^{*2} && \text{degree 3} \\
d_z &= \prod_{\lambda, \mu \in \mathbb{F}_3} (z^* + \lambda y^* + \mu x^*) = z^{*9} + \cdots && \text{degree 9}
\end{aligned}$$

In fact $S^U = k[d_x, d_y, d_z]$.

Example contd.

U acts on the dual space V as $\left\{ \begin{pmatrix} 1 & 0 & 0 \\ * & 1 & 0 \\ * & * & 1 \end{pmatrix} : * \in \mathbb{F}_3 \right\}$, basis x, y, z.

U fixes z.

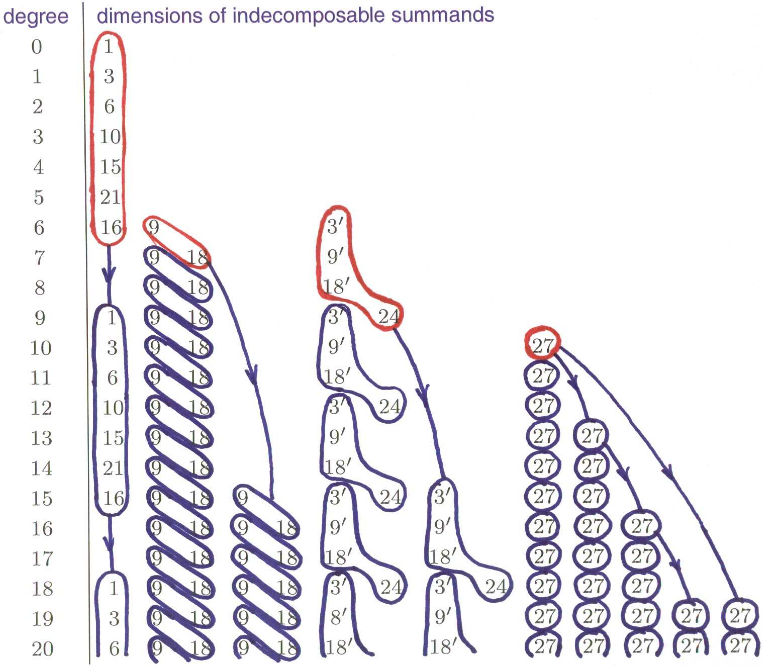
$U_x = \left\{ \begin{pmatrix} 1 & * & * \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} : * \in \mathbb{F}_3 \right\}$ fixes $\langle y, z \rangle$ pointwise.

$U_y = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix} : * \in \mathbb{F}_3 \right\}$ fixes $\langle x, z \rangle$ pointwise.

$U_3(\mathbb{F}_3)$

degree	dimensions of indecomposable summands											
0	1											
1	3											
2	6											
3	10											
4	15											
5	21											
6	$3'$	9	16									
7	9	$9'$	18									
8	9	18	$18'$									
9	1	$3'$	9	18	24							
10	3	9	$9'$	18	27							
11	6	9	18	$18'$	27							
12	$3'$	9	10	18	24	27						
13	9	$9'$	15	18	27	27						
14	9	18	$18'$	21	27	27						
15	$3'$	$3'$	9	9	16	18	24	27	27			
16	9	9	$9'$	$9'$	18	18	27	27	27			
17	9	9	18	18	$18'$	$18'$	27	27	27			
18	1	$3'$	$3'$	9	9	18	18	24	24	27	27	27
19	3	9	9	$9'$	$9'$	18	18	27	27	27	27	27
20	6	9	9	18	18	$18'$	$18'$	27	27	27	27	27

$U_3(\mathbb{F}_3)$


degree
dimensions of indecomposable summands
0 1
1 3
2 6
3 10
4 15
5 21
6 16 9 3′
7 9 18 9′
8 9 18 18′
9 1 9 18 3′ 24
10 3 9 18 9′ 27
11 6 9 18 18′ 27
12 10 9 18 3′ 24 27
13 15 9 18 9′ 27 27
14 21 9 18 18′ 27 27
15 16 9 18 9 3′ 24 3′ 27 27
16 9 18 9 18 9′ 9′ 27 27 27
17 9 18 9 18 18′ 18′ 24 27 27 27
18 1 9 18 9 18 3′ 24 3′ 24 27 27 27
19 3 9 18 9 18 8′ 9′ 27 27 27 27 27
20 6 9 18 9 18 18′ 18′ 27 27 27 27 27

Example contd.

The right hand red block is spread out by three variables. It is projective (relative to the trivial group 1) and 1 fixes the whole 3-dimensional space V.

The second to right red block is spread out by two variables. It is projective relative to U_y, which fixes a 2-dimensional subspace of V.

The second to left red block is spread out by two variables. It is projective relative to U_x, which fixes a 2-dimensional subspace of V.

The left hand red block is spread out by one variable. It is projective relative to U, which fixes a 1-dimensional subspace of V.

Spot the pattern.

Commutative Algebra

Dimension: A finite R-module has **dimension** d if it is **finite** over some polynomial subalgebra $k[a_1, \ldots, a_d] \leq R$, but not for any **smaller** d.

Depth: A finite R-module has **depth** d if it is **free** over some polynomial subalgebra $k[a_1, \ldots, a_d] \leq R$, but not for any **larger** d.

Cohen-Macaulay: depth = dimension; in other words, free and finite over some polynomial subalgebra.

In the picture, all four pieces are Cohen-Macaulay, but this does not hold in general

Hochster: Life is really worth living...in a Cohen-Macaulay ring.

Later revised to: Life is worth living. Period.

Functors

Given a kG-module M, the classical covariants $k[V, M] = (k[V] \otimes M)^G$ are the equivariant polynomial functions from V to M.

In characteristic 0 these are all Cohen Macaulay of dimension $\dim V$ (provided V is faithful).

Notice that $k[V, M] \cong \mathrm{Hom}_{kG}(M^*, k[V])$ and $k[V]^G \cong \mathrm{Hom}_{kG}(k, k[V])$, so we mostly consider functors related to $\mathrm{Hom}_{kG}(M, -)$.

For example, $\widehat{\mathrm{Ext}}^i_{kG}(M, -)$ or $\mathrm{Hom}_{kG}(P_U, -)$, where U is a simple kG-module and P_U is its projective cover. The latter counts the multiplicity of U as a composition factor.

Functors Contd.

We are most interested in indecomposable summands, so we want $\mathrm{Hom}^{\oplus}_{kG}(M, -)$ for indecomposable M.

This can be defined as $\mathrm{Hom}_{kG}(M, -)/\operatorname{rad}\mathrm{Hom}_{kG}(M, -)$, where rad denotes the radical in the category $kG-\mathrm{mod}$.

More concretely, for a finite dimensional indecomposable kG-module M, let

$$J(M, N) = \{f \in \mathrm{Hom}_{kG}(M, N) \quad | \; f \text{ is } \textbf{not} \text{ split injective}\}$$

and define $\mathrm{Hom}^{\oplus}_{kG}(M, N) = \mathrm{Hom}_{kG}(M, N)/J(M, N)$.

Alternatively, $J(M, N)$ is the k-span of the $f \in \mathrm{Hom}_{kG}(M, N)$ that factor through an indecomposable module that is not isomorphic to M.

Functors Contd.

We have:

$\mathrm{Hom}^{\oplus}_{kG}(M, -)$ commutes with $\oplus$.

$\mathrm{Hom}^{\oplus}_{kG}(M, M) = \mathrm{End}_{kG}(M)/\operatorname{rad}\mathrm{End}_{kG}(M) \cong k$ (k algebraically closed.)

$\mathrm{Hom}^{\oplus}_{kG}(M, N) = 0$ if N is indecomposable and not isomorphic to M.

It follows that $\dim_k \mathrm{Hom}^{\oplus}_{kG}(M, X)$ is the number of times that M occurs in a decomposition of X into indecomposable summands.

$\mathrm{Hom}^{\oplus}_{kG}(M, S)$ is a finite S^G-module.

Thus we obtain the multiplicity of M as a summand of S in each degree as the graded dimension of a graded noetherian module.

Define $S^{\oplus G} = \mathrm{Hom}^{\oplus}_{kG}(k, S)$. It is naturally a ring: the ring of trivial summands of S.

The Brauer Construction

For a kG-module M, let $M^{[G]} = M^G / \sum_{p|[G:H]} \mathrm{tr}_H^G M^H$.

For simplicity, assume that G is a p-group.

If X is a G-set and $k[X]$ is the space of functions $X \to k$, considered as a kG-module, then the natural map

$$k[X^G] \to k[X]^{[G]}$$

is an isomorphism.

We will consider $S^{[G]}$. It is naturally a ring and is finite as an S^G-module.

Fixed Point Sets

Given $H \leq G$, define $I_H \leq S$ to be the ideal generated by all elements of the form $(h-1)s$ for $h \in G$, $s \in S$. Let $I = \cap_{H \in \mathrm{Syl}_p(G)} I_H$.

If G is a p-group, then this defines $V^G \leq V$, where V denotes $\mathrm{Spec}(S)$. In general, we get all the points fixed by some Sylow p-subgroup.

The ideal $S^G \cap I \leq S^G$ gives us $V^G \leq V/G$.

Fixed Point Sets Contd.

Theorem The natural homomorphisms of k-algebras

$$S^{[G]} \twoheadrightarrow S^{\oplus G} \twoheadrightarrow S^G/(S^G \cap I) \hookrightarrow (S/I)^G$$

induce universal homeomorphisms on the spectra.

This means that we get homeomorphisms on the spectra that remain homeomorphisms after any base change.

In characteristic p this is the same as what is sometimes called a purely inseparable isogeny and is equivalent to the ring homomorphisms being F-isomorphisms.

This means that the homomorphism of rings has locally nilpotent kernel and for any element in the codomain, some p^n-power is in the image.

Note that $\mathrm{Spec}((S/I)^G) \cong \mathrm{Spec}(S)^P/N_G(P)$, where P is a Sylow p-subgroup.

Fixed Point Sets Contd.

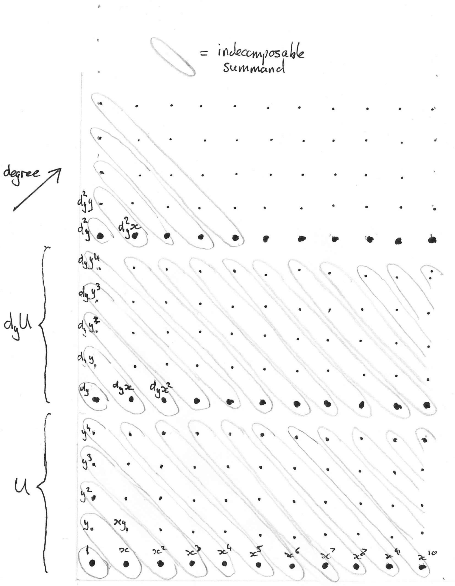
Corollary When G is a p-group and V is a kG-module we have

$$\dim k[V]^{[G]} = \dim k[V]^{\oplus G} = \dim_k V^G.$$

One can also work relative to a different class of subgroups. For example, the trivial subgroup.

Theorem $\mathrm{Spec}(\hat{H}^0(G;S)) \cong (\mathrm{Sing}_p \mathrm{Spec}(S))/G$, where Sing_p means fixed by an element of order p.

Example: Cyclic Group and Two Variables



$G = C_p$ cyclic order p

$S = k[x, y]$

$d_y = y^p - x^{p-1} y$

$S^G = k[x, d_y]$

Action: $y \mapsto y + x$, $x \mapsto x$

k-basis: $\{ x^i y^j d_y^k : 0 \le j \le p-1 \}$

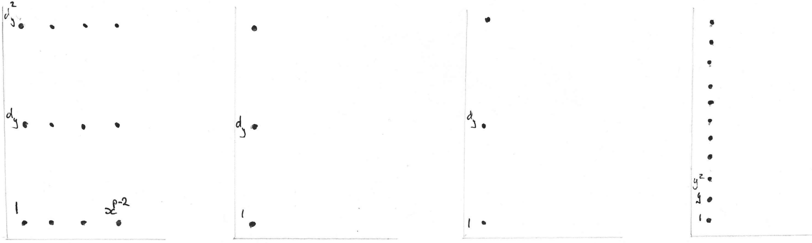
$U =$ span of $\{ x^i y^j : 0 \le j \le p-1 \}$

U is a $k[x]G$ - submodule of S

$S \cong k[d_y] \otimes U$ as a $k[x, d_y]G$-module

$S^G = \operatorname{soc}_{kG} S = k[x, d_y] =$

Example Contd.: Cyclic Group and Two Variables



$k[d_y] \otimes \langle 1, \dots, x^{p-2} \rangle$ $\qquad$ $k[d_y]$ $\qquad$ $k[d_y]$ $\qquad$ $k[y]$

$$k[V]^{[G]} \longrightarrow k[V]^{\oplus G} \longrightarrow k[V^G < V/G] \longrightarrow k[V^G < V]$$

Depth and Dimension

The previous example easily generalises.

Proposition Let G be a p-group, V a kG-module, $\dim_k V = n$, $\dim_k V^G = r$. Then there are elements $d_1, \dots, d_n$ such that:

- $k[V^G]$ is finite over $k[d_1, \dots, d_r]$,
- there is a finite $k[d_{r+1}, \dots, d_n]G$-submodule $U \leq k[V]$,
- $S = k[d_1, \dots, d_r] \otimes_k U$.

Most functors commute with $k[d_1, \dots, d_r] \otimes_k$, so we automatically get depth $\geq r$.

We can leverage this.

Depth and Dimension Contd.

Let M be an indecomposable kG-module with vertex P and source U. The inertia subgroup I is the stabiliser in $N_G(P)$ of the isomorphism class of U. Let I_p denote its Sylow p-subgroup.

Theorem We have

$$\dim \operatorname{Hom}^{\oplus}_{kG}(M, k[V]) \leq \dim V^P.$$

If $\operatorname{Hom}^{\oplus}_{kG}(M, k[V]) \neq 0$, then

$$\operatorname{depth} \operatorname{Hom}^{\oplus}_{kG}(M, k[V]) \geq \dim V^{I_p}.$$

Corollary If $I_p = P$ then $\operatorname{Hom}^{\oplus}_{kG}(M, k[V])$ is Cohen-Macaulay of dimension $\dim_k V^P$, in particular if P is a Sylow p-subgroup. The ring of trivial summands $k[V]^{\oplus G}$ is always Cohen-Macaulay of dimension V^{G_p}.

Depth and Dimension Contd.

$\mathrm{Hom}^{\oplus}_{kG}(M, k[V])$ is not always Cohen-Macaulay.

In characteristic 0, $k[V]^G$ is Gorenstein if $\det(V) = 1$. But even for a p-group, $k[V]^{\oplus G}$ is not always Gorenstein.

If the theorem doesn't hold for one functor, try another.

Theorem $\mathrm{Hom}_k(M, k[V])^{[G]}$ is Cohen Macaulay of dimension $\dim V^P$.

More General Rings

Results about dimension also apply to general S.

Theorem $\dim \operatorname{Hom}^{\oplus}_{kG}(M, S) \leq \dim \operatorname{Spec}(S)^P$.

We can also consider equivariant coherent sheaves. These are just finite graded SG-modules, where SG is the twisted group algebra, or, alternatively, S-modules with a compatible G action.

Theorem $\operatorname{Supp}(\operatorname{Hom}^{\oplus}_{kG}(M, \mathcal{F})) \leq \operatorname{Spec}(S)/G$ is contained in the image of $\operatorname{Supp}(\mathcal{F})^P \leq \operatorname{Spec}(S)$.

More on $k[V]$

Theorem For $k[V]$ only a finite number of isomorphism types of indecomposable modules occur.

This is not true in general.

Theorem We have $\operatorname{reg} \operatorname{Hom}^{\oplus}_{kG}(M, k[V]) \leq 0$.

reg is the Castelnuovo-Mumford regularity.

It follows that if we know that $\operatorname{Hom}^{\oplus}_{kG}(M, k[V])$ is finite over $R = k[d_1, \ldots, d_r] \leq k[V]^G$ then $\operatorname{Hom}^{\oplus}_{kG}(M, k[V])$ has generators and relations as an R-module in degrees at most $\sum(\deg d_i - 1)$.

More is true. If a computer can calculate the multiplicity of M as a summand of S in degrees up to $\sum(\deg d_i - 1)$ then we can deduce the multiplicity in all degrees.

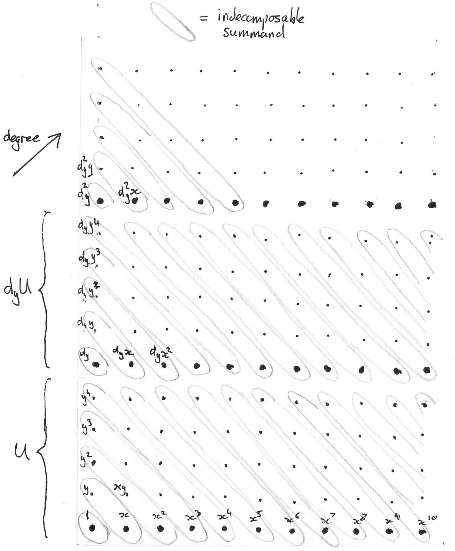
Finite Group Schemes

Theorem Some of this extends to finite groups schemes with *unipotent* identity component.

Avoid vertex. Can use relative projectivity.

The results for fixed points are not true without extra hypotheses on the action.

Example: α_p and Two Variables



$G = \alpha_p$

$S = k[x,y]$

$d_y = y^p$

$S^G = k[x, d_y]$

Action: $x\frac{\partial}{\partial y}$

k-basis: $\{ x^i y^j d_y^k : 0 \le j \le p-1 \}$

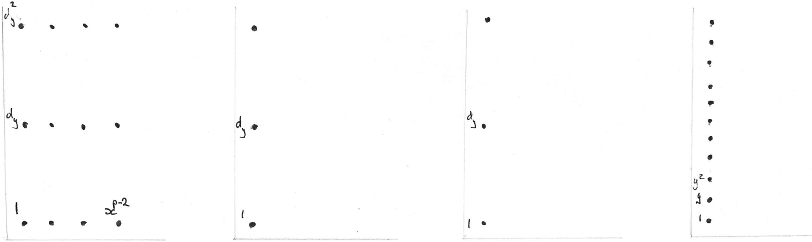
$U =$ span of $\{ x^i y^j : 0 \le j \le p-1 \}$

U is a $k[x]G$-submodule of S

$S \cong k[d_y] \otimes U$ as a $k[x, d_y]G$-module

$S^G = \operatorname{soc}_{kG} S = k[x, d_y] =$ 

Example Contd.: α_p and Two Variables



$k[d_y] \otimes \langle 1, \dots, x^{p-2} \rangle$ $\qquad$ $k[d_y]$ $\qquad$ $k[d_y]$ $\qquad$ $k[y]$

$$k[V]^{[G]} \longrightarrow k[V]^{\oplus G} \longrightarrow k[V^G < V/G] \longrightarrow k[V^G < V]$$

Finite Group Schemes

In general, this approach fails.

For example, for sl_2 acting in the obvious way on $k[x, y]$ infinitely many non-isomorphic indecomposable summands occur.

This seems to be linked to the failure of the Normal Basis Theorem.

Theorem If a finite group G acts on a field K with fixed field K^G then, considered as a $K^G G$-module, K is free of rank 1.

In the case of sl_2 acting on $k(x, y)$, the action is not even projective.

The action of sl_2 on $k[x, y]$ is faithful, but the stabiliser of the generic point is non-trivial.