Blocks of group algebras over local rings

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Blocks

- G: **finite** group
- R: a commutative domain (e.g. $\mathbb{Q}$, $\mathbb{C}$, $\boxed{\mathbb{F}_p, \bar{\mathbb{F}}_p}$,...)

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There is a unique factorisation (up to reordering)

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for indecomposable R-algebras B_i. We call the B_i the **blocks** of RG.

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- We write "$\times$" since this is a direct product of rings. The B_i are also two-sided ideals, and RG is their direct sum "$\oplus$".
- On the level of **representations/module catelgories**:

$$\boxed{\mathrm{Mod}{-}RG \simeq \mathrm{Mod}{-}B_1 \oplus \ldots \oplus \mathrm{Mod}{-}B_\ell}$$

  so it makes sense to consider the representation theory of individual blocks.

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Remark

We usually don't use the word "block" when $R = \mathbb{C}$.
By Maschke's theorem $\mathbb{C}G$ is semisimple for any finite G, so the only possible "blocks" are full matrix algebras $M_n(\mathbb{C})$.

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Modular representation theory

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Example: $G = S_3$, $p = 2$

Let $e = () + (1,2,3) + (1,3,2) \in \bar{\mathbb{F}}_2 S_3$. Then

$$\boxed{\bar{\mathbb{F}}_2 S_3 = \bar{\mathbb{F}}_2 S_3 e \times \bar{\mathbb{F}}_2 S_3 (1 - e) \cong \bar{\mathbb{F}}_2[X]/(X^2) \times M_2(\bar{\mathbb{F}}_2)}$$

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We're interested in representation theory, so we really only care about their module categories.

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Defect groups

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To each block B of kG we attach a p-subgroup $D \leqslant G$, unique up to conjugation, called the **defect group** of B. We say "B has defect D".

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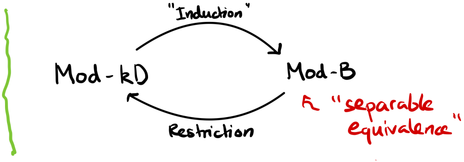

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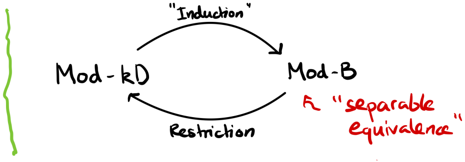

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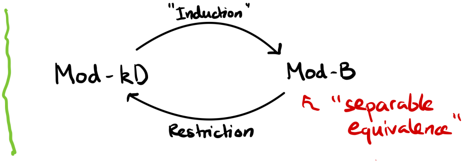

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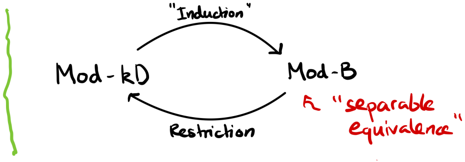

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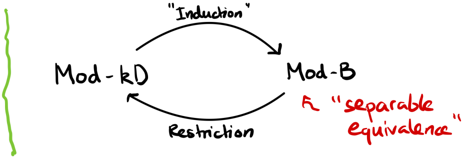

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- Blocks of dihedral, semidihedral and ~~generalised~~ quaternion defect ($p = 2$) are neatly classified (Erdmann).

Donovan's Conjecture

Let D be a p-group. Up to Morita equivalence, there are only finitely many block algebras with defect D.

(we are looking at <u>all</u> group algebras kG for <u>all</u> finite groups G)

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structure of block algebras over k
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- These are not particularly strong restrictions!
- We don't know much else!

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So the answer to the original question is disappointingly "no". Block algebras can be highly irrational.

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- a semisimple $k((t))$-algebra A. In a strong form of the conjecture

$$A = M_{n_1}(k((t))) \times M_{n_2}(k((t))) \times \ldots \times M_{n_\ell}(k((t))),$$

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- a $k[[t]]$-subalgebra $\Lambda \subset A$ such that
  - Λ is finitely generated as a $k[[t]]$-module,
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This is open! (Barnea-Ginosar, 2008, proved that $\bar{\mathbb{F}}_2 Q_8$ is not a counterexample)

A different coefficient ring

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- $\mathcal{O}$ is a commtative domain of characteristic 0.
- $\mathcal{O}$ is local with maximal ideal $p\mathcal{O}$.
- $\mathcal{O}$ is complete w.r.t. the topology with $\{p^i\mathcal{O} \mid i \in \mathbb{N}\}$ open neighbourhoods of 0.
- $\mathcal{O}/p\mathcal{O} \cong k$.

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  This descends to blocks!
- Formally, $k[[t]]$ and $\mathcal{O}$ share many properties, but $\mathcal{O}$ has characteristic zero.

Example: $G = S_3$, $p = 2$

$$KG \cong K \times K \times \begin{pmatrix} K & K \\ K & K \end{pmatrix}$$

$$\cup$$

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$$kG \cong k[X]/(X^2) \times \begin{pmatrix} k & k \\ k & k \end{pmatrix}$$

defect C_2 defect $\{1\}$

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$\equiv_3$ $\quad$ $\equiv_3$

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$$kG \cong k\left(\bullet \underset{\beta}{\overset{\alpha}{\rightleftarrows}} \bullet\right) / (\alpha\beta\alpha,\ \beta\alpha\beta)$$

defect C_3

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This played a role in the reduction result Donovan's conjecture for abelian defect mentioned before (which required to work over $\mathcal{O}$).

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Theorem (E, 2021)

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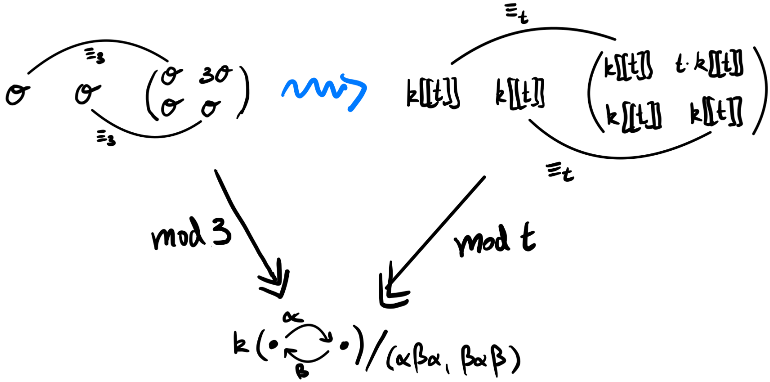
- More is true: if Λ is a block algebra, then there are only finitely many lattices L of any given rank with $\mathrm{Ext}^1(L, L) = 0$ ("rigid" lattices).
  This includes permutation lattices, but is a purely representation theoretic property.

The proof would be easy if we were working over $k[[t]]$ instead of $\mathcal{O}$ . . .



How a proof would look over $k[[t]]$ instead of $\mathcal{O}$

Take a $k[[t]]$-order Λ in a semisimple $k((t))$-algebra A. Assume $HH^1(\Lambda) = 0$.

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The previous argument ultimately carries over to $\mathcal{O} = W(k)$.

What exactly is $W(k)$?

As a set, we can define $W(k) = k[[t]]$, with multiplication and addition

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- An easy consequence is that $\mathrm{GL}_n(\mathcal{O}/p^m\mathcal{O})$ is an algebraic group over k.

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Works over $\mathcal{O}$ basically as is

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becomes $\mathrm{Aut}(W(k[\varepsilon]/\varepsilon^2) \otimes_{W(k)} \Lambda)$,

use $k[[x^{1/p^\infty}]]$ instead of $k[\varepsilon]/\varepsilon^2$.

Thank you for your attention!