Some problems on modular group algebras

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Groups and their actions
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Tensor factorizations

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Definition

A has the tensor Krull-Schmidt property if whenever

- $A \cong \bigotimes_{i=1}^{n} A_i \cong \bigotimes_{j=1}^{m} B_j$, and
- each of the A_i's and B_j's is tensor indecomposable,

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This is not the case in general:

$$\mathbb{C} \otimes_{\mathbb{R}} \mathbb{H} \cong \mathbb{C} \otimes_{\mathbb{R}} M_2(\mathbb{R}),$$

where $\mathbb{H}$ is the ring of real quaternions.

Tensor factorizations of local algebras

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- Let $\mathbb{L}_k$ be the class of local augmented k-algebras.

Theorem (Horst, 1987)

Suppose that

1. $char(k) = 0$,
2. $N \otimes R \cong N \otimes S$ *in* $\mathbb{L}_k$, *and*
3. R *is noetherian and* N *artinan.*

Then $R \cong S$.

Tensor factorizations of local algebras

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Suppose that

1. $char(k) = 0$,
2. $A \in \mathbb{L}_k$ *is noetherian.*

Then there is a decomposition

$$A \cong B \otimes A_1 \otimes \cdots \otimes A_n$$

such that

1. *each* A_i *is tensor indecomposable, and*
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Moreover, this decomposition is unique up to isomorphism and reordering.

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No analogue is known when $\mathrm{char}(p) > 0$.

Tensor factorizations of local group algebras

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Question

Does kG have the Krull-Schmidt property in $\mathbb{L}_k$?

Tensor factorizations of local commutative group algebras

Theorem (Carlson-Kovacs, 1995)

Suppose that

1. *G abelian finite p-group, and*
2. *$kG = A_1 \otimes A_2$.*

Then $G = G_1 \times G_2$ such that $A_i \cong kG_i$ for each i.

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Questions on modular p-group algebras

Let's drop the commutativity assumption.

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- *Does every directly indecomposable group have tensor indecomposable group algebra over k?*

Tensor factorizations with a commutative factor

Does every tensor factorization of kG come from a direct decomposition of G?

Theorem (GL-del Río-Sakurai, in progress)

Suppose that

1. $\mathbb{F}_p G = A_1 \otimes A_2$, *and*
2. A_1 *is commutative.*

Then $G = G_1 \times G_2$ *and* $A_i \cong \mathbb{F}_p G_i$ *for each* i.

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Then $G = G_1 \times G_2$ *and* $A_i \cong \mathbb{F}_p G_i$ *for each* i.

This implies that $\mathbb{F}_p G$ admits a unique decomposition $A_1 \otimes A_2$ with A_1 commutative and A_2 without commutative tensor factors.

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Corollary (GL, del Río, Sakurai)

Suppose that G is indecomposable and

- *either G can be generated by 3 elements, or*
- *G' is cyclic.*

Then $\mathbb{F}_p G$ is indecomposable.

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Question (Modular Isomorphism Problem)

Can the isomorphism type of G be recovered from $\mathbb{F}_p G$?

Answer:

- If $p = 2$, it can't (GL-Margolis-del Río, 2022).
- If $p > 2$, we do not know.

The modular isomorphism problem

Question (Modular Isomorphism Problem)

Can the isomorphism type of G be recovered from $\mathbb{F}_p G$?

Yes, provided that one of the following holds:

- G is abelian (Deskins, 1956).
- G is metacyclic (Bagiński 1988, Sandling 1996).
- $\gamma_2(G)^p \gamma_3(G) = 1$ (Sandling, 1989).
- ...

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These results both fail when $p = 2$.

Thanks for your attention.