From extension of groups to realization-obstruction of graded algebras and back

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Multiplying two such elements we obtain a 2-place function $\alpha : G \times G \to N$ such that

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as well as an automorphism $\eta_g \in$Aut(N) for every $g \in G$ such that

$$\eta_g(n) = u_g n u_g^{-1} =: \widehat{u_g}(n).$$

N abelian

If N is abelian then $\eta : \begin{array}{ccc} G & \to & \mathrm{Aut}(N) \\ g & \mapsto & \eta_g \end{array}$ is a group homomorphism, and $\alpha : G \times G \to N$ is a 2-cocycle, where N is a G-module via this action η.

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The extension is said to be **of type** $\bar{\eta}$.

Developing the associativity assumption $u_{g_1}(u_{g_2}u_{g_3}) = (u_{g_1}u_{g_2})u_{g_3}$ for every $g_1, g_2, g_3 \in G$ we have

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This action is shown to be free and transitive, yielding a (non-canonical) 1-1 correspondence between these sets. It induces a free and transitive action of the cohomology group $H^2_\eta(G, \mathcal{Z}(N))$ on the set of type-$\bar{\eta}$ extensions up to equivalence.

When $\eta : G \to \mathrm{Aut}(N)$ is already a homomorphism (even before moding out by Inn(N)), e.g. when N is abelian, there is a distinguished extension of type $\bar{\eta}$, namely the **semidirect product** $N \rtimes_\eta G$.

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In fact, it is not even clear if there is *any* extension of type $\bar{\eta}$.
Question. How to check that?

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$$(\diamondsuit) \;\; \widehat{\beta(g, hk)}^{-1} \circ \eta(g)\widehat{(\beta(h, k))}^{-1} \circ \widehat{\beta(g, h)} \circ \widehat{\beta(gh, k)} = \mathrm{Id}_N.$$

This brings us back to the non-abelian coboundary

$$\partial\beta(g,h,k) := \beta(g,hk)^{-1} \cdot \eta(g)(\beta(h,k))^{-1} \cdot \beta(g,h) \cdot \beta(gh,k).$$

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Moreover, it turns out that this class is also independent of the lifting η of $\bar{\eta} \in \mathrm{Hom}(G, \mathrm{Out}(N))$.

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Theorem (Eilenberg-MacLane)

An outer action $\bar{\eta} \in Hom(G, Out(N))$ gives rise to a well-defined cohomology class $[c]_{\bar{\eta}} \in H^3_\eta(G, \mathcal{Z}(N))$, which serves as an obstruction to the existence of an extension of type $\bar{\eta}$.

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Example: If $\gcd(|G|, |\mathcal{Z}(N)|) = 1$ then $H^*_\eta(G, \mathcal{Z}(N)) = 1$ no matter what the G-module structure of $\mathcal{Z}(N)$ is. Thus, for every $\bar{\eta} \in \mathrm{Hom}(G, \mathrm{Out}(N))$ there is a unique extension of type $\bar{\eta}$.

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such that

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Moreover, every homogeneous component A_g is an A_e-bimodule and $(\heartsuit)$ describes a decomposition of A as a direct sum of A_e-bimodules.

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(b) Adding an action $\phi \in \text{Hom}(\mathbb{Z}, \text{Aut}_k(K))$ yields the corresponding **skew Laurent polynomial algebra** $K[X^{\pm 1}; \phi]$, which is a crossed product as well.

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In particular, the group algebra with respect to the group of integers $G = \mathbb{Z}$ can be identified with the Laurent polynomial algebra $K[X^{\pm 1}]$.
(b) Adding an action $\phi \in \mathrm{Hom}(\mathbb{Z}, \mathrm{Aut}_k(K))$ yields the corresponding **skew Laurent polynomial algebra** $K[X^{\pm 1}; \phi]$, which is a crossed product as well.
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Note that the $\mathbb{Z}$-graded algebras $K[X]$ and, more generally, $K[X; \phi]$ are not crossed products.
(c) If k is a field, then every Brauer similarity class in $\mathrm{Br}(k)$ can be represented by a **classical crossed product** $K * G$, where K is a Galois field extension of k, and $G = \mathrm{Gal}[K : k]$.

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If A_e is commutative, then this automorphism does not depend on the choice of u_g as above, and the rule

$$\Phi : \begin{array}{ccc} G & \to & \mathrm{Aut}_k(A_e) \\ g & \mapsto & \widehat{u_g} \end{array}$$

is a group homomorphism, that is a G-action on A_e.

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When A_e is not necessarily commutative, then exactly as in the group setting, distinct choices of invertible elements result in A_e-automorphisms which differ by inner A_e-automorphisms. There is a well-defined outer action (a **collective character**)

$$\Phi : \begin{array}{ccc} G & \to & Out_k(A_e) = \mathrm{Aut}_k(A_e)/\mathrm{Inn}(A_e) \\ g & \mapsto & \widehat{u_g} \cdot \mathrm{Inn}(A_e) \end{array} .$$

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Then these sets can be (non-canonically) identified.

Strongly graded algebras

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Then any homogeneous component A_g, or rather its isomorphism class $[A_g]$, is **invertible** as an A_e-bimodule since

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These indeed generalize collective characters since

$$\mathrm{Out}_k(A_e) < \mathrm{Pic}_k(A_e).$$

Another generalization is a morphism $\mathrm{Pic}_k(R) \xrightarrow{h} \mathrm{Aut}_k(\mathcal{Z}(R))$ (which splits if the k-algebra R is commutative), associating to any GCC $\Phi : G \to \mathrm{Pic}_k(A_e)$, a G-module structure on the abelian group $\mathcal{Z}(A_e^*)$, and, in turn, an obstruction class

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If Φ is unobstructed, then the strongly graded algebras of type Φ (up to equivalence) are similarly identified with $H^2_{\Phi_0}(G, \mathcal{Z}(A_e^*))$.

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$$(*)\quad 1 \to H^2_{\Phi_0}(G, K^*) \to \mathit{Cliff}_k(\Phi_0) \to \mathcal{C}_k(\Phi_0) \to H^3_{\Phi_0}(G, K^*).$$

Next, $\mathrm{Pic}(K)$ acts on both terms $\mathrm{Cliff}_k(\Phi_0)$ [Haefner - del Río] and $\mathcal{C}_k(\Phi_0)$ respecting both skew products, in a way such that

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The sequence can be completed as follows

Theorem (A.Antony-Y.G.)

Let K be a commutative k-algebra. Then there is a 7-term exact sequence of abelian groups

$$\begin{aligned}1\ \ &\to H^1_{\phi_0}(G,K^*) \to Ext(\phi_0, Pic(K)) \to Pic(K)^{\phi_0} \to H^2_{\phi_0}(G,K^*)\\ &\to Cliff_k(\phi_0)/Pic(K) \to C_k(\phi_0)/Pic(K) \to H^3_{\phi_0}(G,K^*).\end{aligned}$$

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Other generalizations were suggested by Kanzaki, Miyashita, El Kaoutit and J. Gómez-Torrecillas, Dokuchaev et. al.

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as a disjoint union of subsets indexed by the elements of G (**cosets**) such that $S_e \cong R$, and

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It turns out that the addition and so the direct sum in ($\heartsuit$) are somehow redundant.

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$$(\blacklozenge) \quad S = \sqcup_{g \in G} S_g$$

as a disjoint union of subsets indexed by the elements of G (**cosets**) such that $S_e \cong R$, and

$$S_g \cdot S_h := \{x_g \cdot x_h | \ x_g \in S_g, x_h \in S_h\} \subseteq S_{g \cdot h}, \quad \forall g, h \in G.$$

It turns out that the addition and so the direct sum in ($\heartsuit$) are somehow redundant.

Observation (A.Antony-Y.G.)

One operation is needed to develop the theory, including Picard groups of invertible bisets, and a corresponding 7-term sequence for semigroup extensions.

Thanks for your attention.