Linear degenerations of Schubert varieties via quiver Grassmannians

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Groups and their actions

Levico Terme, June 2024

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(Borel subgroup)

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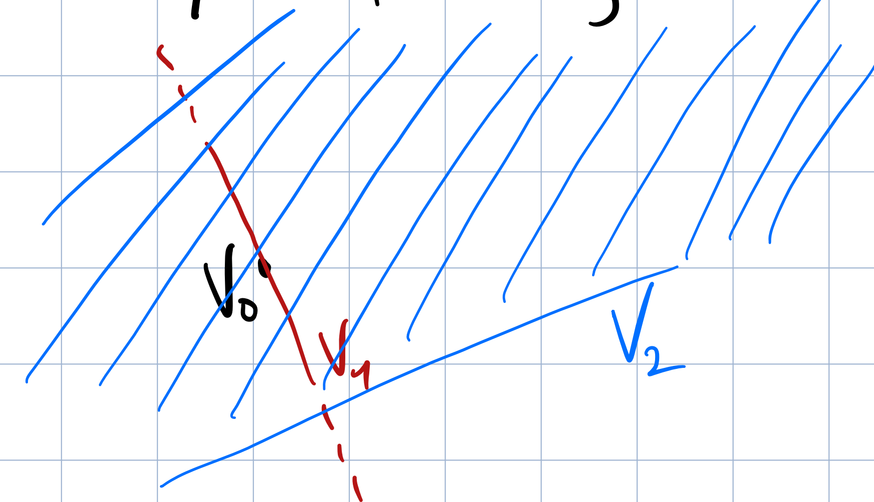

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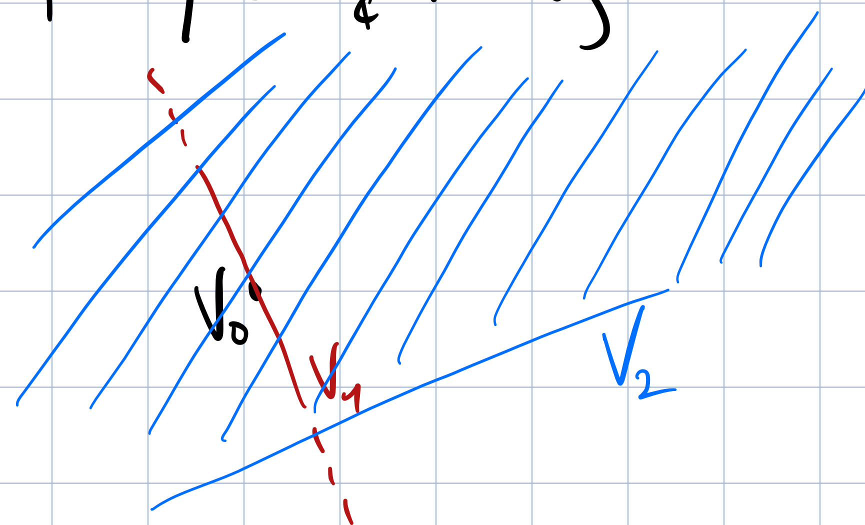

- Fl_{n+1} is a smooth, complex, projective algebraic variety
- GL_{n+1} acts transitively on Fl_{n+1} (base change)

$\longrightarrow$ We consider instead the action of B on Fl_{n+1}

The action of B on Fl_{n+1} yields finitely many orbits, or cells, indexed by the elements w of S_{n+1}.

These cells $\underline{C}_w$ are isomorphic to affine spaces and form a stratification of Fl_{n+1}.

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$$X_w := \overline{C_w} \quad \text{(Zariski closure)}$$

- $X_w = \bigcup\limits_{\substack{v \in S_{n+1} \\ v \leq w}} C_v$, where "$\leq$" is Bruhat order in S_{n+1}

Ex: Bruhat order in S_3:

$$\begin{tikzcd}
& {[321]} \arrow[dl, dash] \arrow[dr, dash] & \\
{[231]} \arrow[d, dash] \arrow[drr, dash] & & {[312]} \arrow[d, dash] \arrow[dll, dash] \\
{[213]} \arrow[dr, dash] & & {[132]} \arrow[dl, dash] \\
& {[123]} &
\end{tikzcd}$$

Quiver Grassmannians

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EX: Consider A_n quiver, fix an A_n-representation M:

$$\begin{array}{ccccccccc} 1 & & 2 & & 3 & & & & n \\ \bullet & \xrightarrow{id} & \bullet & \xrightarrow{id} & \bullet & \xrightarrow{id} & \cdots & \xrightarrow{id} & \bullet \\ \mathbb{C}^{n+1} & & \mathbb{C}^{n+1} & & \mathbb{C}^{n+1} & & & & \mathbb{C}^{n+1} \end{array}$$

and fix dimension vector $\underline{e} = (1, 2, \dots, n) \in \mathbb{Z}^n_{\geq 0}$.

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The quiver Grassmannian $Gr_{\underline{e}}(M)$ is the moduli space of subrepresentations of M with dimension vector $\underline{e}$.

In this example: $Gr_{\underline{e}}(M) = \{ V_1 \subset V_2 \subset \ldots \subset V_n \mid \dim V_i = i \}$

$\rightsquigarrow$ $Gr_{\underline{e}}(M) \cong Fl_{n+1}$

and then more generally $\rightsquigarrow$

Linear degenerations of flag varieties

(Cerulli Irelli, Fang, Feigin, Fourier, Reineke 2016)

$$\underset{\mathbb{C}^{n+1}}{\overset{1}{\bullet}} \xrightarrow{f_1} \underset{\mathbb{C}^{n+1}}{\overset{2}{\bullet}} \xrightarrow{f_2} \underset{\mathbb{C}^{n+1}}{\overset{3}{\bullet}} \xrightarrow{f_3} \cdots \xrightarrow{f_{n-1}} \underset{\mathbb{C}^{n+1}}{\overset{n}{\bullet}}, \quad f_* \in \prod_{i=1}^{n-1} \mathrm{End}(\mathbb{C}^{n+1}), \quad \underline{e} = (1, \dots, n)$$

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- The group $G := \prod_{i=1}^{n} GL_{n+1}$ with elements $g_* = (g_1, \dots, g_n)$ acts on $R := \prod_{i=1}^{n-1} \mathrm{End}(\mathbb{C}^{n+1})$: $g_* \cdot f_* := (g_2 f_1 g_1^{-1}, \dots, g_n f_{n-1} g_{n-1}^{-1})$

and $\left\{ \begin{array}{c} \text{ORBITS} \\ \mathcal{O}_M \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{ISOMORPHISM CLASSES OF REPRESENTATIONS} \\ M \end{array} \right\}$.

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- **Def:** <u>M degenerates to N</u> if $N \in \overline{\mathcal{O}_M}$ $\quad (\mathcal{O}_N \subset \overline{\mathcal{O}_M})$

The orbits $\mathcal{O}_M$ and the relations among their closures are described by <u>RANK TUPLES</u>:

if $M = (f_1, \ldots, f_{n-1})$, $\underline{r}^M := (r_{ij}^M)_{i<j}$ where $r_{ij}^M := \mathrm{rank}(f_{j-1} \circ \cdots \circ f_i)$

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Ex:

$$\mathbb{C}^3 \xrightarrow{\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} =: M} \mathbb{C}^3 \qquad \mathbb{C}^3 \xrightarrow{\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} =: N} \mathbb{C}^3$$

$$\underline{r}^M = (3), \quad \underline{r}^N = (2)$$

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$$\underline{r}^M = (3), \ \underline{r}^N = (2)$$

$\rightsquigarrow$ if $n+1=3$, the (partial) order on the rank tuples is:

$$\begin{array}{c}(3)\\ |\\ (2)\\ |\\ (1)\\ |\\ (0)\end{array}$$

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\begin{tikzcd}
\mathbb{C} \arrow[r, "id"] \arrow[d, "\iota_1"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathbb{C} \arrow[r, "id"] \arrow[d, "\iota_1"] \arrow[dr, phantom, "\circlearrowleft"] & \mathbb{C} \arrow[r, dashed] \arrow[d] & \cdots \arrow[r, "id"] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & \mathbb{C} \arrow[d, "\iota_1"] \\
\mathbb{C}^2 \arrow[r] \arrow[d, "\iota_2"] & \bullet \arrow[r] \arrow[d, "\iota_2"] & \bullet \arrow[r, dashed] \arrow[d] & \cdots \arrow[r] \arrow[d] & \mathbb{C}^2 \arrow[d, "\iota_2"] \\
\vdots & \vdots & \vdots & & \vdots \\
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with $\iota_k : \mathbb{C}^k \hookrightarrow \mathbb{C}^{k+1}$

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$\rightsquigarrow$ choosing appropriate dimension vector $\underline{e}^w$:

$$\underline{X_w \cong Gr_{\underline{e}^w}(M)}$$

$\rightsquigarrow$ Replace $(id, \dots, id)$ by $f_* = (f_1, \dots, f_{n+1})$: $f_i : \mathbb{C}^{n+1} \longrightarrow \mathbb{C}^{n+1}$
upper triangular

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The representation M^{f_*} is:

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$\rightsquigarrow$ Replace $(\mathrm{id}, \dots, \mathrm{id})$ by $f_* = (f_1, \dots, f_{n-1})$: $f_i : \mathbb{C}^{m+1} \longrightarrow \mathbb{C}^{m+1}$
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$$\rightsquigarrow \underline{X_w^{f_*} := \mathrm{Gr}_{\underline{e}_w}(M^{f_*})}$$

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$\rightsquigarrow$ Theorem:

- $|\{\text{orbits } \mathcal{O}_M\}| < \infty$
- $\exists$ combinatorial description:

$$\mathcal{O}_N \subset \overline{\mathcal{O}_M} \quad \text{iff} \quad \underline{r}^N \leq \underline{r}^M$$

where $\underline{r}^M, \underline{r}^N$ are "like" rank tuples

Ex:

$$\begin{tikzcd}
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$\underline{r}^M = (1,1,2,1,2,3)$ $\qquad$ $\underline{r}^N = (1,1,2,0,1,2)$

Ranks of all non-trivial south-west minors

Here $\underline{r}^N \leq \underline{r}^M \Rightarrow \mathcal{O}_N \subset \overline{\mathcal{O}_M}$.

Thank you!