Do we need all Sylow subgroups?

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Notation

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Notations

Let G be a group and let p be a prime. Then

- A subgroup $P \leq G$ is a Sylow p-subgroup of G if p does not divide $|G : P|$.
- An element $x \in G$ is a p-element if $o(x) = p^a$ for some $a \geq 0$.
- G_p denotes the set of p-elements of G.
- $\mathrm{Syl}_p(G)$ denotes the set of Sylow p-subgroups of G.

Redundant Sylow p-subgroups

It is easy to see that

$$\bigcup_{P \in \mathrm{Syl}_p(G)} P = G_p$$

Question

Do we need all Sylow p-subgroups to cover G_p?

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Question

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The answer is **NO**. However, the answer is yes more often than one could perhaps expect.

Definition

We say that G possesses **redundant Sylow p-subgroups** if we do not need all Sylow p-subgroups to cover G_p.

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(II) A_9 possesses redundant Sylow 2-subgroups.

(III) Both $\mathrm{SL}(2, q)$ and $\mathrm{PSL}(2, q)$ posses redundant Sylow 2-subgroups for $q \in \{11, 13, 19, 23, 25, 27, 29, 37, \ldots\}$.

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(III) Both $\mathrm{SL}(2,q)$ and $\mathrm{PSL}(2,q)$ posses redundant Sylow 2-subgroups for $q \in \{11, 13, 19, 23, 25, 27, 29, 37, \ldots\}$.

(IV) $\mathrm{PSL}(3,7)$ possesses redundant Sylow 3-subgroups.

In view of these examples, it seems to be no pattern for the existence of redundant Sylow p-subgroups.
Now, the natural questions that arise are the following.

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Which groups possess a redundant Sylow p-subgroup?

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Which p-groups can appear as the Sylow subgroup of a group with a redundant Sylow p-subgroup?

Alternative approach

Proposition

G does not possess redundant Sylow p-subgroups if and only if there exists $x \in G_p$ such that x lies in a unique Sylow p-subgroup.

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Proof.

Let $\mathrm{Syl}_p(G) = \{P_1, P_2, \ldots, P_n\}$. *The result is clear for* $n = 1$. *Assume that* $n > 1$. *The group G has a redundant Sylow p-subgroup if and only if* $G_p = \bigcup_{i \neq j} P_i$ *for some* j. *This happens if and only if* $P_j \subseteq \bigcup_{i \neq j} P_i$, *which is equivalent to saying that every element of* P_j *lies in more than one Sylow p-subgroup.* □

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- If G has cyclic Sylow p-subgroups, then G does not possesses redundant Sylow p-subgroups.
- If $P \cap Q = 1$ for every $P, Q \in \mathrm{Syl}_p(G)$ with $P \neq Q$, then G does not possesses redundant Sylow p-subgroups.

Theorem (Navarro)

Let G be a finite group. Suppose that x belongs to a unique Sylow p-subgroup P of G. Then $C_G(x) \subseteq N_G(P)$.

Proof.

Let $c \in C_G(x)$. Then $x^c = x \in P$, so $x \in P^{c^{-1}}$. Since P is the unique Sylow p-subgroup that contains x, we deduce that $P = P^{c^{-1}}$, so $c \in N_G(P)$, as wanted. □

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Let M be the Monster Group $P \in \mathrm{Syl}_7(M)$. We know that 5 divides $|C_G(x)|$ for any $x \in P$, but $|N_G(P)|$ is not divisible by 5. Thus, M possesses redundant Sylow 7-subgroups.

The following was pointed out to us by Thomas Weigel.

Theorem

Let G be a group of Lie type in characteristic p. Then there exists $x \in G_p$ such that x lies in a unique Borel subgroup of G. In particular, x lies in a Sylow p-subgroup.

As a consequence the groups of Lie type in characteristic p do not possess redundant Sylow p-subgroups.

Constructions of groups with redundant Sylow

Theorem (Maróti, M. and Moretó, 2024 and Sambale, 2024)

Let p be a prime, and let P be a non-cyclic p-group. There exists a **solvable** *group G such that possesses redundant Sylow p-subgroups and the Sylow subgroups of G are isomorphic to P.*

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Remark

In the original result, we had the extra hypothesis "P **has exponent** p". This hypothesis was removed by Sambale.

Some comments about the groups in the above theorem.

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- In this case, having redundant Sylow subgroups is equivalent to satisfying $C_N(P) = 1$ and $C_N(x) > 1$ for all $x \in P$.
- The original construction and Sambale's construction differ in the way of choosing the complement N.
- In fact, we can produce examples with arbitrarily large Fitting height.

Constructions of groups with redundant Sylow

Given p a prime, we also have examples of non-solvable groups with redundant Sylow p-subgroups.

Theorem (Maróti, M. and Moretó, 2024)

Let p be an odd prime and let q be a prime power such that p divides $q-1$ but p^2 does not divide $q-1$. Then

(i) *If $1 < n < p$, then $\mathrm{GL}(n,q)$ does not posses redundant Sylow p-subgroups.*

(ii) $\mathrm{GL}(p,q)$ *possesses a redundant Sylow p-subgroup.*

Remark

We observe that such a prime power q exists by Dirichlet's Theorem.

Some special cases

Theorem (Maróti, M. and Moretó, 2024)

Let $n \geq 2$. Then S_n does not possess redundant Sylow p-subgroups for any prime p. In particular, if $p > 2$ then A_n does not possess redundant Sylow p-subgroups.

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Proof.

Let $n = \sum_{i=0}^{k} a_i p^i$ be the p-adic expansion of n. Let $x \in S_n$ be an element with a_i cycles of length p^i for each i. Then, x lies in a unique Sylow p-subgroup of S_n □

Examples

Let $n = 15$ and $p = 2$. Then $15 = 2^3 + 2^2 + 2^1 + 1$ the element

$$(1, 2, 3, 4, 5, 6, 7, 8)(9, 10, 11, 12)(13, 14)(15)$$

lies in a unique Sylow 2-subgroup of S_{15}.

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Let $n = 15$ and $p = 3$. Then $15 = 3^2 + 3^1 + 3^1$ the element

$$(1, 2, 3, 4, 5, 6, 7, 8, 9)(10, 11, 12)(13, 14, 15)$$

lies in a unique Sylow 3-subgroup of S_{15}.

Question

What happens with A_n when $p = 2$?

Theorem (Maróti, M. and Moretó, 2024)

Let $n \geq 6$ with $n = \sum_{i=r}^{k} a_i 2^i$, where $a_r, a_{r+1}, \dots, a_k \in \{0,1\}$, $a_r = a_k = 1$. The group A_n has a redundant Sylow 2-subgroup if and only if the following conditions are satisfied:

- $\sum_{i=1}^{k} a_i \equiv 1 \pmod 2$ *if n is odd.*
- *$r \geq 2$ is even and $\sum_{i=r}^{k} a_i \equiv 1 \pmod 2$ if n is even.*

Thus, A_9 and A_{16} are the first alternating groups with redundant Sylow 2-subgroups.

In fact, it was a consequence of the following result.

Theorem

Let $n \geq 6$ and let $x \in (S_n)_2$. Then x lies in a unique Sylow 2-subgroup of S_n if and only if x has at most two fixed points and all cycles of x of length bigger than one have different lengths.

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Assume that $n = 8$. Then $8 = 2^3$ and

$$(1, 2, 3, 4, 5, 6, 7, 8) \notin A_8$$

but $8 - 2 = 2^2 + 2^1$ and

$$(1, 2, 3, 4)(5, 6)(7)(8) \in A_8$$

examples

Assume that $n = 9$. Then $9 = 2^3 + 1$ and

$$(1, 2, 3, 4, 5, 6, 7, 8)(9) \notin A_9$$

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$$(1, 2, 3, 4, 5, 6, 7, 8)(9) \notin A_9$$

Assume that $n = 16$. Then $16 = 2^4$ and

$$(1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16) \notin A_{16}$$

moreover $16 - 2 = 2^3 + 2^2 + 2^1$ and

$$(1, 2, 3, 4, 5, 6, 7, 8)(9, 10, 11, 12)(13, 14)(15)(16) \notin A_{16}$$

Theorem (Maróti, M. and Moretó, 2024)

The group $\mathrm{PSL}(2, q)$ *possesses redundant Sylow p-subgroups if and only if* $p = 2$ *and* q *is none of the following.*

a) $q = 2^k$ *for an integer* k.

b) $q = 2^k + 1$ *for an integer* k.

c) $q = 2^k - 1$ *for an integer* k.

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c) $q = 2^k - 1$ *for an integer* k.

Remark

(I) If p divides q, then the Sylow p-subgroups of $\mathrm{PSL}(2, q)$ intersect trivially.

(II) If p is odd and does not divide q, then $\mathrm{PSL}(2, q)$ possesses cyclic Sylow p-subgroups.

In both cases, $\mathrm{PSL}(2, q)$ does not possess redundant Sylow p-subgroups.

Thank you for your attention.

Any question?