06/04/24

Continuum Braid Group

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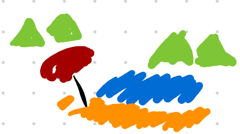


Outline

1. Motivation
2. Artin Braid group
3. Birman Ko Lee Presentation
4. Continuum Braid group of the interval
5. Affinization

Motivation 1

$$\curvearrowleft \hat{\mathcal{B}}^{e}_{\mathrm{rank}\,\mathfrak{g}+1} = \langle \hat{T}_{\omega_i} \rangle$$

$$\mathcal{U}_q^{Dr}(\hat{\mathfrak{g}}) \xleftrightarrow{\simeq} \mathcal{U}_q^{DJ}(\hat{\mathfrak{g}})$$

$x^+_{i,r}$ $x^-_{i,r}$ $i = 1, \ldots, \mathrm{rank}\,\mathfrak{g}$
$+ R$ $r \in \mathbb{Z}$

E_i F_i K_i $i = 1, \ldots, \mathrm{rank}\,\mathfrak{g}+1$ $+ R$

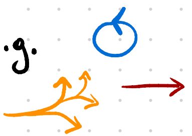
$$x^-_{i,r} \longleftarrow \hat{T}^{\kappa}_{\omega_i}(F_i)$$

1

Motivation 2

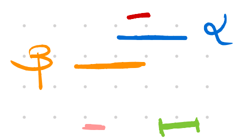
Cartan Datum: X C.Q $\qquad U_q(X)$ gen $E_\alpha \; F_\alpha \; K_\alpha^{\pm 1} \qquad \alpha \subseteq X$

E.g.



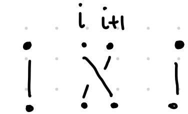
E.g. $X = \longrightarrow$

$$[E_\alpha, F_\beta] = \delta_{\alpha\beta} \frac{K_\alpha - K_\alpha^{-1}}{q - q^{-1}} + a_{\alpha\beta}\left(q^{c^+_{\alpha\beta}} E_{\alpha\setminus\beta} K_\beta^{a_{\alpha\beta}} - q^{c^-_{\alpha\beta}} K_\alpha F_{\beta\setminus\alpha} \right)$$
$$+ \, b_{\alpha\beta} \, q^{b_{\alpha\beta}} (q - q^{-1}) \, E_{(\alpha\cup\beta)\setminus\beta} \, K_{\alpha\cap\beta}^{b_{\alpha\beta}} \, F_{(\alpha\cup\beta)\setminus\beta}$$



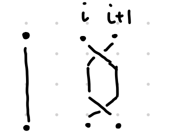
Braid group B_{N+1}

Generators s_i $\quad i=1 \quad N$ + Relations BS $s_i s_{i+1} s_i = s_i s_{i+1} s_i$;
BL $s_i s_j = s_j s_i$ $\quad |i-j| \neq 1$



$= s_i$

PB_{N+1} Pure Braid subgroup of B_{N+1} : Generators $a_i = s_i$
+ Relations



$= a_i$

$W_{N+1} = B_{N+1} / PB_{N+1}$ $\quad G: s_i \quad i=1, \quad ,N$ + R: BS; BL; $s_i^2 = 1$

Length function $W_{N+1} \ni g \Rightarrow \ell(g) = r$ if r minimal s.t. $g = s_{i_1} \quad s_{i_r}$ reduced expression

THEOREM [M; I-T]

$g = \sigma_{i_1} \quad \sigma_{i_r} = \sigma_{j_1} \quad \sigma_{j_r}$; $W_{N+1} \longrightarrow B_{N+1}$ $\Rightarrow S_{i_1} \quad S_{i_r} = S_{j_1} \quad S_{j_r}$
reduced $\qquad \sigma_{i_1} \quad \sigma_{i_r} \longrightarrow s_{i_1} \quad s_{i_r}$ $\qquad$ in B_{N+1}

Birman Ko Lee Presentation B_{N+1}

$\mathcal{G}: a_{rs}\ 1\le r<s\le N+1$ $\ \mathcal{R}:$ S $\ a_{st}\, a_{qr} = a_{rq}\, a_{ts} \qquad (t-r)(t-q)(s-r)(s-q)>0$

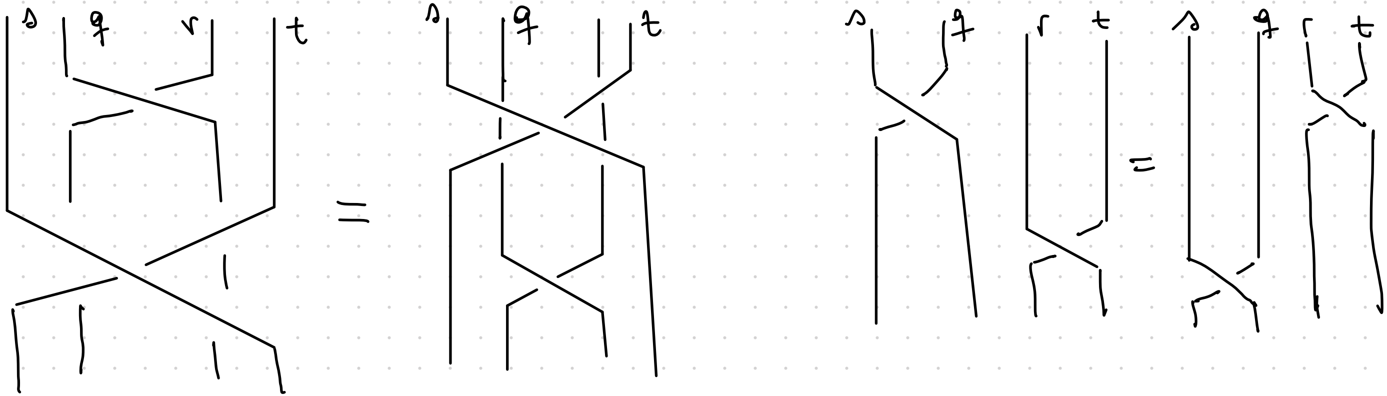
L $\ a_{ts}\, a_{sr} = a_{rt}\, a_{st} = a_{rs}\, a_{rt} \qquad 1\le r<s<t\le N+1$

remark

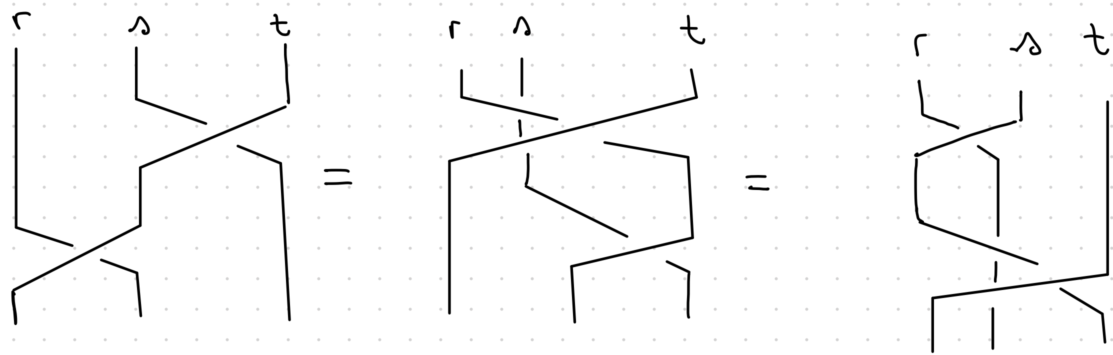
S asserts that a_{st} and a_{qr} commute if t & s do not separate q & r

L expresses a type of "partial" commutativity if a_{st} & a_{qr} share a common strand, namely if one consider $a_{ts}\, a_{sr}$ one can move a_{st} to the right (resp a_{st} to the left) at the expense of increasing (resp. decreasing) the first (resp the second) subscript of a_{rs} to t (resp of a_{sr} to r).

S



L



Continuum Braid Group

BKL R & G does not depend on the set of $\{1 \le r < s \le N\}$ only on their mutual position.

$$\mathcal{J} := \Big\{ \{0 < x_1 < \ \ < x_N < 1\} = \underline{x} \Big\} \ ; \quad I \ni \underline{x}, \underline{y} \qquad \underline{x} \preceq \underline{y} \iff \underline{x} \subseteq \underline{y}$$

$$\mathcal{B}_{\underline{x}} = \Big\{ g : S_{x_i} \ \Big|\ x_i \in \underline{x} \ + \ S \ \& \ L \Big\} \cong \mathcal{B}_{|\underline{x}|}$$

Proposition given $\underline{x} \preceq \underline{y} \preceq \underline{z}$

$\underline{1}$ One can define

$$\mathcal{B}_{\underline{x}} \xrightarrow{\varphi_{\underline{x}\underline{y}}} \mathcal{B}_{\underline{y}} \quad \text{st:} \quad \varphi_{\underline{x}\underline{x}} = \mathrm{id} \quad \&$$

$$\begin{tikzcd}
\mathcal{B}_{\underline{x}} \arrow[r, "\varphi_{\underline{x}\underline{y}}"] \arrow[dr, "\varphi_{\underline{x}\underline{z}}"'] & \mathcal{B}_{\underline{y}} \arrow[d, "\varphi_{\underline{y}\underline{z}}"] \\
 & \mathcal{B}_{\underline{z}}
\end{tikzcd}$$

2 The same holds if consider to $\mathcal{PB}_{\underline{x}}$ & $W_{\underline{x}}$

3
$$\begin{tikzcd}
\mathcal{B}_I \arrow[r, phantom, ":="] \arrow[rr, bend left] & \varprojlim \mathcal{B}_x \arrow[r, "\circlearrowleft"] & \varprojlim W_{\underline{x}} := W_{\underline{I}} \\
\mathcal{PB}_I \arrow[u, hook] \arrow[r, phantom, ":="] & \varprojlim \mathcal{PB}_x \arrow[u, hook] \arrow[ul, phantom, "\circlearrowright"] &
\end{tikzcd}$$

Problem: ☹ consider $\forall e \neq g \in W_{\underline{x}_1} \Rightarrow \exists \{\underline{x}_i\} \in \mathcal{J}$

$\underline{x}_1 \leq \underline{x}_2 \leq \underline{x}_n \ldots$ st $\lim_{i \to +\infty} \ell_i(g) = +\infty$

Solution: absolute length λ, that is

$g \in W_{\underline{x}}$ $\quad \lambda(g) = r$ if r minimal s.t. $g = a_{p_1 q_1} \quad a_{p_r q}$

Proposition

$$\lambda_{\underline{x}}(g) = \lambda_{\underline{y}}(g) \quad \forall \underline{x} \leq \underline{y}$$

corollary

$\lambda_I(g) = \lambda_{\underline{x}}(g)$ is well defined in W_I

THEOREM

$g \in W_I \qquad g = \alpha_{p_1 q_1} \cdots \alpha_{p_r q_r}$ reduced

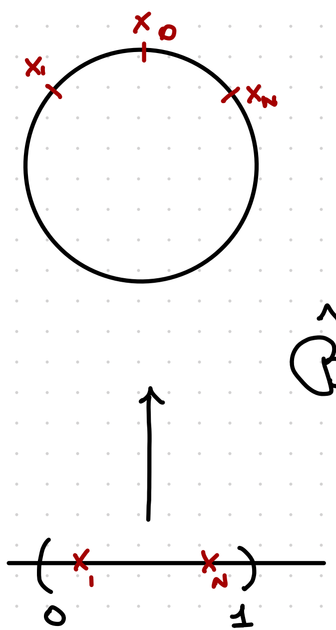
$\qquad\qquad\qquad = \alpha_{m_1 n_1} \cdots \alpha_{m_r n_r}$

consider $W_I \xrightarrow{\mathcal{M}} \mathcal{B}_I$

$\alpha_{p_1, q_1} \cdots \alpha_{p_r q_r} \longmapsto a_{p_1 q_1} \cdots a_{p_r q_r}$

$\Rightarrow \mathcal{M}(\alpha_{p_1 q_1} \cdots \alpha_{p_r q_r}) = \mathcal{M}(\alpha_{m_1 n_1} \cdots \alpha_{m_r n_r})$ $\square$

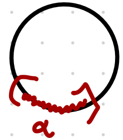
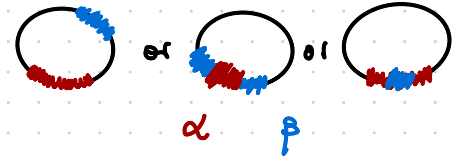
Generalization



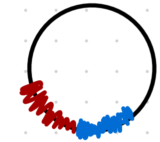
Why stop to the interval I?

$$\hat{\mathcal{B}}_{\underline{x}} = \mathfrak{g} = \{ s_{x_i} \mid i = 0 \quad N \} + R \qquad S\ L \quad (\underline{\text{modulo } N})$$

$$\hat{\mathcal{B}}_I := \mathcal{B}_{S^1} = \varinjlim \mathcal{B}_{\underline{x}} \qquad (\text{same construction})$$

$\mathcal{B}_{S^1}$ $\mathcal{G}: S_\alpha$ st  + Rel $S_\alpha S_\beta = S_\beta S_\alpha$ if 

S

$S_\beta S_\alpha = S_{\alpha \sqcup \beta} S_\beta = S_\alpha S_{\alpha \sqcup \beta}$ if 

L

$S_\alpha S_\beta$ no rel if $\alpha = S^1 \setminus \beta$

$A_1^{(1)}$

$\mathcal{B}^{e}_{S^1}$ $\mathcal{G}: \{S_\alpha ; T_x \mid \alpha \subseteq S^1 ; x \in \mathbb{R}\}$

$\mathcal{R}$ $\{S_\alpha$ S L $A_1^{(1)}$ $T_x T_y = T_{x+y}$ $T_x S_\alpha = S_{x(\alpha)} T_x$

where

$T_x((a,b]) \longrightarrow (a+x, b+x]$

$\mathcal{U}_q(\rightarrow) \curvearrowleft \mathcal{B}_I$

$E_\alpha \quad F_\alpha \quad K_\alpha^{\pm 1} \quad \alpha \in \rightarrow$

$\rightsquigarrow$

$\mathcal{U}_q(\hat{\rightarrow}) \curvearrowleft \mathcal{B}^e_{S^1}$

$E_{\alpha,k} \quad F_{\alpha k} \quad K_{\alpha,k}^{\pm 1}$

$k \in \mathbb{Z} \quad \alpha \subseteq \rightarrow$

$\mathcal{U}_q(\circlearrowleft) \curvearrowleft \mathcal{B}_{S^1}$

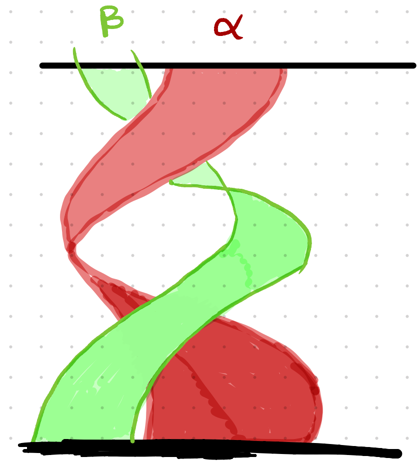
$E_\alpha \quad F_\alpha \quad K_\alpha^{\pm} \quad \alpha \subseteq S^1$

$\longleftrightarrow$

THEOREM

this is an isomorphism

Grazie!



any questions?