Graded relations on crossed products

Yuval Ginosar and Ofir Schnabel

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Groups and their actions: algebraic, geometric and combinatorial aspects

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- A G-grading is a **crossed product** if A_g admits an invertible element for any $g \in G$.

Examples of gradings

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- Notice that $\psi\big|_R$, the restriction of ψ to R is in $\text{Aut}_F(R)$.

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- Two G-Clifford system extensions (A, ι_1), (A, ι_2) of R are **graded isometric** if there exists a graded equivalence (ψ, φ) with $\psi : A \widetilde{\to} A$, $\varphi \in$Aut(G) such that $\psi \circ \iota_1 = \iota_2$.

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- Two G-Clifford system extensions (A, ι_1), (A, ι_2) of R are **equivalent as Clifford systems** if there exists a graded isomorphism $\psi : A \xrightarrow{\sim} A$ such that $\psi \circ \iota_1 = \iota_2$.

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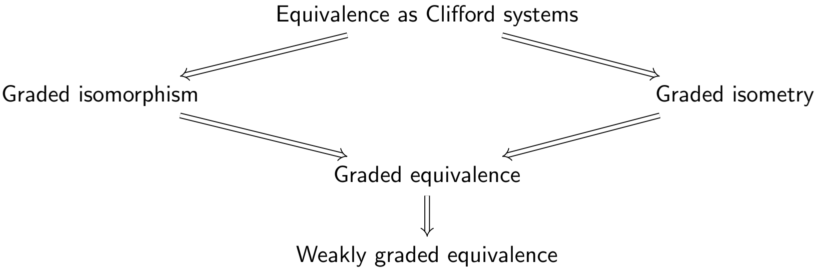
Theorem

For Crossed products the notion of graded equivalence and weakly graded equivalence coincide.

Diagram of equivalence relations

We add the following diagram to help understand the order of coarsening between the different equivalence relations on graded algebras.



Where these equivalences appear?

- Graded isomorphism- The graded Artin-Wedderburn Theorem.
- Graded equivalence- The intrinsic fundamental group $\pi_1(A)$ of an algebra A.
- Weak graded equivalence- Graded ideals, graded subspaces, or graded polynomial identities.
- Graded isometry- In coding theory in which crossed products are used as ambient spaces where G is finite and R is a finite field.
- Equivalence as Clifford systems- realization-obstruction exact sequences.

Definition of a crossed product, η and α

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- As stated above a G-grading is a **crossed product** if A_g admits an invertible element for any $g \in G$.
- By its definition a crossed product is a free $R = A_e$ module. A choice of a basis of invertible homogeneous elements $\{u_g\}_{g \in G}$ determines:
  1. An outer action $\eta : G \to \mathrm{Aut}_F(R)$ by
  $$\eta(g)(r) := u_g r u_g^{-1}$$
  for $g \in G$ and $r \in R$.
  2. A map (an η-twisting) $\alpha : G \times G \to R^*$ by
  $$\alpha(g, h) := u_g u_h u_{gh}^{-1}$$
  for any $g, h \in G$.
- By choosing $u_e = 1$, we may always assume for any $g \in G$ that $\alpha(1, g) = 1 = \alpha(g, 1)$.

The set Γ

- The associativity of a crossed product yields
$$\alpha(g_1, g_2)\alpha(g_1g_2, g_3) = \eta(g_1)(\alpha(g_2g_3))\alpha(g_1, g_2g_3) \quad (1)$$
and comparing $u_{g_1}u_{g_2}ru_{g_2}^{-1}u_{g_1}^{-1}$ with $u_{g_1g_2}ru_{g_1g_2}^{-1}$ yields
$$(\eta(g_1)\circ\eta(g_2))(r) = \alpha(g_1, g_2)(\eta(g_1g_2)(r))\alpha(g_1, g_2)^{-1}. \quad (2)$$
for any $g_1, g_2, g_3 \in G$, and $r \in R$.

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for any $g_1,g_2,g_3\in G$, and $r\in R$.
- For a group G and an algebra R we define

$$\Gamma := \Gamma_{G,R} = \{(\alpha,\eta) | \text{ s.t. } \alpha \text{ and } \eta \text{ satisfying } (1) \text{ and } (2)\}.$$

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- For $(\alpha,\eta)\in\Gamma$ we denote the corresponding crossed product by $R_\eta^\alpha G$.

Remark

With the above notation, if additionally R is commutative then η is an action and $\alpha\in Z_\eta^2(G,R^*)$ is a 2-cocycle.

Example of crossed products

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- For η trivial, for any value of $u_g^2 = \alpha(g,g) \in \mathbb{C}^*$ we get that $\mathbb{C}^{\alpha}G \cong \mathbb{C}G = \mathbb{C} \oplus \mathbb{C}$.
- For η non-trivial, $u_g^2 = \alpha(g,g) \in \mathbb{R}^*$. There are two cases:
    1. For $u_g^2 = \alpha(g,g) > 0$ we have $\mathbb{C}_{\eta}^{\alpha}G \cong M_2(\mathbb{R})$
    2. For $u_g^2 = \alpha(g,g) < 0$ we have $\mathbb{C}_{\eta}^{\alpha}G \cong \mathbb{H}$ the real Quaternions algebra.

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Remark

When addressing a crossed product $R^{\alpha}_{\eta}G$ with basis $\{u_g\}_{g\in G}$ as a G-Clifford extension of R we mean with respect to the embedding $r \mapsto r \cdot 1$ for any $r \in R$.

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Corollary

Two crossed products $R^{\alpha_1}_{\eta_1}G$ and $R^{\alpha_2}_{\eta_2}G$ are

1. *equivalent as Clifford systems if and only if there exist a graded isomorphism $\psi : R^{\alpha_1}_{\eta_1}G \to R^{\alpha_2}_{\eta_2}G$ such that $\psi\big|_R \in Aut_F(R)$ is trivial.*
2. *R-isometric if and only if there exists a graded equivalence $(\psi, \varphi) : R^{\alpha_1}_{\eta_1}G \to R^{\alpha_2}_{\eta_2}G$ such that $\psi\big|_R \in Aut_F(R)$ is trivial.*

The group K acting on Γ

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- A main tool for that is a group

$$K := K_{G,R} = (R^*)^G \rtimes (\mathrm{Aut}_F(R) \times \mathrm{Aut}(G))$$

acting on Γ where here the group $(R^*)^G$ of functions from G to R^* is with pointwise multiplication.

Main Theorem

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Main Theorem

For an algebra R and a group G, the group K is acting on Γ such that two crossed products $R_\eta^\alpha G$ and $R_{\eta'}^{\alpha'} G$

1. *are equivalent as Clifford systems if and only if (α, η) and (α', η') are in the same $(R^*)^G$ orbit.*
2. *are graded isomorphic if and only if (α, η) and (α', η') are in the same $(R^*)^G \rtimes \mathrm{Aut}_F(R)$ orbit.*
3. *are graded isometric if and only if (α, η) and (α', η') are in the same $(R^*)^G \rtimes \mathrm{Aut}(G)$ orbit.*
4. *are graded equivalent if and only if (α, η) and (α', η') are in the same K orbit.*

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- Using Theorem A we can classify in similar terms and with additional term of the Brauer group all finite dimensional graded division algebras up to the above mention equivalence relations.

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Then $L = Z(D)$ the center of D is either $\mathbb{R}$ or $\mathbb{C}$.

1. For $L = \mathbb{R}$, D is either $\mathbb{R}$ or $\mathbb{H}$. For $D = \mathbb{R}$, η is trivial, and for $D = \mathbb{H}$, by Skolem-Noether Theorem, η is again trivial
2. For $L = \mathbb{C}$, D is also $\mathbb{C}$. We have two automorphisms, the trivial automorphism and the complex conjugation. Hence, the kernel N of η is either G itself or a subgroup of index 2.

Then, we have the following corollary.

Corollary (see also Karrer (1973) and Lavit (2018))

For a graded $\mathbb{R}$-division algebra with a base algebra D over a finite group G, D is either $\mathbb{R}, \mathbb{H}$ or $\mathbb{C}$ and

1. *For $D = \mathbb{R}$ the graded isomorphism classes are exactly the twisted group algebras $\mathbb{R}^{\alpha}G$ where α runs over the distinct classes in $H^2(G, \mathbb{R}^*)$.*
2. *For $D = \mathbb{H}$, again there is a 1-1 correspondence between the graded isomorphism classes and the distinct classes in $H^2(G, \mathbb{R}^*)$.*
3. *For $D = \mathbb{C}$, the kernel N of η is either G or a subgroup of index 2. For each N there is a 1-1 correspondence between the graded isomorphism classes and the distinct sets $\{[\alpha], [\bar{\alpha}]\}$ in $H^2_{\eta}(G, \mathbb{C}^*)$.*

Example

Let $G \cong C_4 \times C_4 = \langle g \rangle \times \langle h \rangle$ and let η be trivial. Consider two cohomology classes $[\alpha], [\overline{\alpha}] \in H^2(G, \mathbb{C}^*)$ defined by

$$[\alpha] : u_g^4 = u_h^4 = 1, u_g u_h = i u_h u_g,$$
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2. are $\mathbb{C}$-graded equivalent.
3. are $\mathbb{R}$-graded isomorphic.
4. are not $\mathbb{R}$-equivalent as Clifford systems.

Example

Let $G = C_4 \times C_4$ and let $F = \mathbb{R}$.

- For the case $L = \mathbb{R}$ we have for both $D = \mathbb{R}$ and $D = \mathbb{H}$ that the G-graded isomorphism classes are in one to one correspondence with $H^2(G, \mathbb{R}^*) \cong C_2 \times C_2 \times C_2$.

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- For $L = \mathbb{C} = D$, we have trivial automorphism and complex conjugation. Denote $N = \ker(\eta)$.
  1. For $N = G$, η is trivial and
  $$H^2_\eta(G, \mathbb{C}^*) = H^2(G, \mathbb{C}^*) \cong C_4 = \langle[\alpha]\rangle$$
  where $[\alpha]$ defined as in the previous example. In this case, there are 3 $\mathbb{R}$-graded isomorphism classes, $\{\mathbb{C}G\}$, $\{\mathbb{C}^{\alpha^2}G\}$, $\{\mathbb{C}^{\alpha}G, \mathbb{C}^{\alpha^3}G\}$.

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  where $[\alpha]$ defined as in the previous example. In this case, there are 3 $\mathbb{R}$-graded isomorphism classes, $\{\mathbb{C}G\}$, $\{\mathbb{C}^{\alpha^2}G\}$, $\{\mathbb{C}^{\alpha}G, \mathbb{C}^{\alpha^3}G\}$.
  2. For η non-trivial, $H^2_\eta(G, \mathbb{C}) \cong C_2 \times C_2$. Therefore, for any action η with kernel N of index 2 there are 4 $\mathbb{R}$-graded isomorphism classes.

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- For $L = \mathbb{C} = D$ we have again two automorphisms, the trivial automorphism and the complex conjugation. Denote $N = \ker(\eta)$.
  1. For $N = G$, η is trivial and $H^2_\eta(G, \mathbb{C}^*) = H^2(G, \mathbb{C}^*)$ is trivial. So, in this case there is only one $\mathbb{R}$-graded isomorphism class, the group algebra $\mathbb{C}G$.
  2. For N the unique subgroup of index 2, $H^2_\eta(G, \mathbb{C}^*)$ is cyclic of order 2 so here there are two $\mathbb{R}$-graded isomorphism classes.

Thank you.