Multiplicity-free induced characters of symmetric groups

Pavel Turek

Royal Holloway, University of London

June 04, 2024

Based on arXiv:2309.07761
To appear in Transactions of the AMS

Outline

Outline

Multiplicity-free characters

Let G be a finite group and V a finite-dimensional $\mathbb{C}G$-module.

Multiplicity-free characters

Let G be a finite group and V a finite-dimensional $\mathbb{C}G$-module. The *character* of V is a (class) function $\rho : G \to \mathbb{C}$ given by the traces of the G-action on V.

Multiplicity-free characters

Let G be a finite group and V a finite-dimensional $\mathbb{C}G$-module. The *character* of V is a (class) function $\rho : G \to \mathbb{C}$ given by the traces of the G-action on V. We adapt the following notions to characters:

Multiplicity-free characters

Let G be a finite group and V a finite-dimensional $\mathbb{C}G$-module. The *character* of V is a (class) function $\rho : G \to \mathbb{C}$ given by the traces of the G-action on V. We adapt the following notions to characters:

- direct sum and tensor products,

Multiplicity-free characters

Let G be a finite group and V a finite-dimensional $\mathbb{C}G$-module. The *character* of V is a (class) function $\rho : G \to \mathbb{C}$ given by the traces of the G-action on V. We adapt the following notions to characters:

- direct sum and tensor products,
- irreducibility and decomposition,

Multiplicity-free characters

Let G be a finite group and V a finite-dimensional $\mathbb{C}G$-module. The *character* of V is a (class) function $\rho : G \to \mathbb{C}$ given by the traces of the G-action on V. We adapt the following notions to characters:

- direct sum and tensor products,
- irreducibility and decomposition,
- induction and restriction.

Multiplicity-free characters

Let G be a finite group and V a finite-dimensional $\mathbb{C}G$-module. The *character* of V is a (class) function $\rho : G \to \mathbb{C}$ given by the traces of the G-action on V. We adapt the following notions to characters:

- direct sum and tensor products,
- irreducibility and decomposition,
- induction and restriction.

We say that a character is *multiplicity-free* if it has no irreducible constituent of multiplicity more than one.

Multiplicity-free characters

Let G be a finite group and V a finite-dimensional $\mathbb{C}G$-module. The *character* of V is a (class) function $\rho : G \to \mathbb{C}$ given by the traces of the G-action on V. We adapt the following notions to characters:

- direct sum and tensor products,
- irreducibility and decomposition,
- induction and restriction.

We say that a character is *multiplicity-free* if it has no irreducible constituent of multiplicity more than one.

Example: Any irreducible character is multiplicity-free.

Multiplicity-free characters

Let G be a finite group and V a finite-dimensional $\mathbb{C}G$-module. The *character* of V is a (class) function $\rho : G \to \mathbb{C}$ given by the traces of the G-action on V. We adapt the following notions to characters:

- direct sum and tensor products,
- irreducibility and decomposition,
- induction and restriction.

We say that a character is *multiplicity-free* if it has no irreducible constituent of multiplicity more than one.

Example: Any irreducible character is multiplicity-free. The character of the regular representation of G is multiplicity-free if and only if G is abelian.

Multiplicity-free characters

Let G be a finite group and V a finite-dimensional $\mathbb{C}G$-module. The *character* of V is a (class) function $\rho : G \to \mathbb{C}$ given by the traces of the G-action on V. We adapt the following notions to characters:

- direct sum and tensor products,
- irreducibility and decomposition,
- induction and restriction.

We say that a character is *multiplicity-free* if it has no irreducible constituent of multiplicity more than one.

Example: Any irreducible character is multiplicity-free. The character of the regular representation of G is multiplicity-free if and only if G is abelian. The natural permutation characters of symmetric groups are multiplicity-free.

Setting

Fix a non-negative integer n.

Setting

Fix a non-negative integer n. Given a character ρ of a subgroup $G \leq S_n$, we say ρ is *induced-multiplicity-free* if $\rho\big\uparrow_G^{S_n}$ is multiplicity-free.

Setting

Fix a non-negative integer n. Given a character ρ of a subgroup $G \leq S_n$, we say ρ is *induced-multiplicity-free* if $\rho\big\uparrow_G^{S_n}$ is multiplicity-free. A subgroup $G \leq S_n$ is *multiplicity-free* if there is an induced-multiplicity-free character of G.

Setting

Fix a non-negative integer n. Given a character ρ of a subgroup $G \leq S_n$, we say ρ is *induced-multiplicity-free* if $\rho\big\uparrow_G^{S_n}$ is multiplicity-free. A subgroup $G \leq S_n$ is *multiplicity-free* if there is an induced-multiplicity-free character of G.

Example: The groups S_n and A_n are multiplicity-free.

Setting

Fix a non-negative integer n. Given a character ρ of a subgroup $G \leq S_n$, we say ρ is *induced-multiplicity-free* if $\rho\uparrow_G^{S_n}$ is multiplicity-free. A subgroup $G \leq S_n$ is *multiplicity-free* if there is an induced-multiplicity-free character of G.

Example: The groups S_n and A_n are multiplicity-free. The trivial group is multiplicity-free if and only if $n \leq 2$.

Setting

Fix a non-negative integer n. Given a character ρ of a subgroup $G \leq S_n$, we say ρ is *induced-multiplicity-free* if $\rho\uparrow_G^{S_n}$ is multiplicity-free. A subgroup $G \leq S_n$ is *multiplicity-free* if there is an induced-multiplicity-free character of G.

Example: The groups S_n and A_n are multiplicity-free. The trivial group is multiplicity-free if and only if $n \leq 2$.

Q1: What are the multiplicity-free subgroups of S_n?

Setting

Fix a non-negative integer n. Given a character ρ of a subgroup $G \leq S_n$, we say ρ is *induced-multiplicity-free* if $\rho\uparrow_G^{S_n}$ is multiplicity-free. A subgroup $G \leq S_n$ is *multiplicity-free* if there is an induced-multiplicity-free character of G.

Example: The groups S_n and A_n are multiplicity-free. The trivial group is multiplicity-free if and only if $n \leq 2$.

Q1: What are the multiplicity-free subgroups of S_n?
Q2: For each multiplicity-free subgroup $G \leq S_n$ what are the irreducible induced-multiplicity-free characters of G?

Multiplicity-free characters of S_n in literature

The classification of multiplicity-free permutation characters of S_n - Wildon in 2009 (for $n \geq 66$) and independently Godsil and Meagher in 2010.

Multiplicity-free characters of S_n in literature

The classification of multiplicity-free permutation characters of S_n - Wildon in 2009 (for $n \geq 66$) and independently Godsil and Meagher in 2010. These are *Gelfand pairs* (S_n, G).

Multiplicity-free characters of S_n in literature

The classification of multiplicity-free permutation characters of S_n - Wildon in 2009 (for $n \geq 66$) and independently Godsil and Meagher in 2010. These are *Gelfand pairs* (S_n, G).

The classification of subgroups $G \leq S_n$ with all irreducible characters of G being induced-multiplicity-free - Anderson, Humphries and Nicholson in 2021.

Multiplicity-free characters of S_n in literature

The classification of multiplicity-free permutation characters of S_n - Wildon in 2009 (for $n \geq 66$) and independently Godsil and Meagher in 2010. These are *Gelfand pairs* (S_n, G).

The classification of subgroups $G \leq S_n$ with all irreducible characters of G being induced-multiplicity-free - Anderson, Humphries and Nicholson in 2021. These are *strong Gelfand pairs* (S_n, G).

Multiplicity-free characters of S_n in literature

The classification of multiplicity-free permutation characters of S_n - Wildon in 2009 (for $n \geq 66$) and independently Godsil and Meagher in 2010. These are *Gelfand pairs* (S_n, G).

The classification of subgroups $G \leq S_n$ with all irreducible characters of G being induced-multiplicity-free - Anderson, Humphries and Nicholson in 2021. These are *strong Gelfand pairs* (S_n, G).

Our **Q1** asks for 'weak Gelfand pairs' (S_n, G).

Multiplicity-free characters of S_n in literature

The classification of multiplicity-free permutation characters of S_n - Wildon in 2009 (for $n \geq 66$) and independently Godsil and Meagher in 2010. These are *Gelfand pairs* (S_n, G).

The classification of subgroups $G \leq S_n$ with all irreducible characters of G being induced-multiplicity-free - Anderson, Humphries and Nicholson in 2021. These are *strong Gelfand pairs* (S_n, G).

Our **Q1** asks for 'weak Gelfand pairs' (S_n, G).

There are related classifications of various multiplicity-free symmetric functions:

- products of Schur functions - Stembridge 2001,

Multiplicity-free characters of S_n in literature

The classification of multiplicity-free permutation characters of S_n - Wildon in 2009 (for $n \geq 66$) and independently Godsil and Meagher in 2010. These are *Gelfand pairs* (S_n, G).
The classification of subgroups $G \leq S_n$ with all irreducible characters of G being induced-multiplicity-free - Anderson, Humphries and Nicholson in 2021. These are *strong Gelfand pairs* (S_n, G).
Our **Q1** asks for 'weak Gelfand pairs' (S_n, G).

There are related classifications of various multiplicity-free symmetric functions:

- products of Schur functions - Stembridge 2001,
- plethysms of Schur functions - Bessenrodt, Bowman and Paget 2022,

Multiplicity-free characters of S_n in literature

The classification of multiplicity-free permutation characters of S_n - Wildon in 2009 (for $n \geq 66$) and independently Godsil and Meagher in 2010. These are *Gelfand pairs* (S_n, G).

The classification of subgroups $G \leq S_n$ with all irreducible characters of G being induced-multiplicity-free - Anderson, Humphries and Nicholson in 2021. These are *strong Gelfand pairs* (S_n, G).

Our **Q1** asks for 'weak Gelfand pairs' (S_n, G).

There are related classifications of various multiplicity-free symmetric functions:

- products of Schur functions - Stembridge 2001,
- plethysms of Schur functions - Bessenrodt, Bowman and Paget 2022,
- combined products $s_\lambda(s_\nu \circ s_\mu)$ - T. 2023.

Outline

Partitions

A *partition* is a non-increasing sequence $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_t)$ of positive integers with *size* $|\lambda| := \sum_{i=1}^{t} \lambda_i$.

Partitions

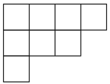
A *partition* is a non-increasing sequence $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_t)$ of positive integers with *size* $|\lambda| := \sum_{i=1}^{t} \lambda_i$. We call the elements λ_i *parts* and say λ is a partition of n and write $\lambda \vdash n$ if $|\lambda| = n$.

Partitions

A *partition* is a non-increasing sequence $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_t)$ of positive integers with *size* $|\lambda| := \sum_{i=1}^{t} \lambda_i$. We call the elements λ_i *parts* and say λ is a partition of n and write $\lambda \vdash n$ if $|\lambda| = n$.

Example: The Young diagram of the partition $(4, 3, 1)$

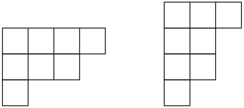
.



Partitions

A *partition* is a non-increasing sequence $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_t)$ of positive integers with *size* $|\lambda| := \sum_{i=1}^{t} \lambda_i$. We call the elements λ_i *parts* and say λ is a partition of n and write $\lambda \vdash n$ if $|\lambda| = n$.

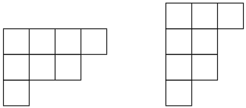
Example: The Young diagrams of the partition $(4, 3, 1)$ and its *conjugate partition* $(3, 2, 2, 1)$.



Partitions

A *partition* is a non-increasing sequence $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_t)$ of positive integers with *size* $|\lambda| := \sum_{i=1}^{t} \lambda_i$. We call the elements λ_i *parts* and say λ is a partition of n and write $\lambda \vdash n$ if $|\lambda| = n$.

Example: The Young diagrams of the partition $(4, 3, 1)$ and its conjugate partition $(3, 2, 2, 1)$ often denoted as $(3, 2^2, 1)$.



Characters of symmetric groups

The irreducible characters of the symmetric group S_n, commonly denoted by χ^λ, are labelled by partitions λ of n.

Characters of symmetric groups

The irreducible characters of the symmetric group S_n, commonly denoted by χ^λ, are labelled by partitions λ of n.
We will often use the identity $\chi^\lambda \times \mathrm{sgn} = \chi^{\lambda'}$.

Characters of symmetric groups

The irreducible characters of the symmetric group S_n, commonly denoted by χ^λ, are labelled by partitions λ of n.
We will often use the identity $\chi^\lambda \times \mathrm{sgn} = \chi^{\lambda'}$.

Example: The characters $\chi^{(n)}$ and $\chi^{(1^n)}$ are the trivial character and the sign of S_n, respectively.

Characters of symmetric groups

The irreducible characters of the symmetric group S_n, commonly denoted by χ^λ, are labelled by partitions λ of n.
We will often use the identity $\chi^\lambda \times \mathrm{sgn} = \chi^{\lambda'}$.

Example: The characters $\chi^{(n)}$ and $\chi^{(1^n)}$ are the trivial character and the sign of S_n, respectively. The natural permutation character of S_n decomposes as $\chi^{(n)} + \chi^{(n-1,1)}$.

Characters of symmetric groups

The irreducible characters of the symmetric group S_n, commonly denoted by χ^λ, are labelled by partitions λ of n.
We will often use the identity $\chi^\lambda \times \mathrm{sgn} = \chi^{\lambda'}$.

Example: The characters $\chi^{(n)}$ and $\chi^{(1^n)}$ are the trivial character and the sign of S_n, respectively. The natural permutation character of S_n decomposes as $\chi^{(n)} + \chi^{(n-1,1)}$.

The irreducible characters of $S_k \times S_l$ are $\chi^\mu \boxtimes \chi^\nu$ where $\mu \vdash k$ and $\nu \vdash l$.

Characters of symmetric groups

The irreducible characters of the symmetric group S_n, commonly denoted by χ^λ, are labelled by partitions λ of n.
We will often use the identity $\chi^\lambda \times \mathrm{sgn} = \chi^{\lambda'}$.

Example: The characters $\chi^{(n)}$ and $\chi^{(1^n)}$ are the trivial character and the sign of S_n, respectively. The natural permutation character of S_n decomposes as $\chi^{(n)} + \chi^{(n-1,1)}$.

The irreducible characters of $S_k \times S_l$ are $\chi^\mu \boxtimes \chi^\nu$ where $\mu \vdash k$ and $\nu \vdash l$.
The irreducible characters of $S_m \wr S_h$ are constructed from the *elementary irreducible characters* $\chi^\mu \wr \chi^\nu := \widetilde{(\chi^\mu)^{\times h}} \chi^\nu$ where $\mu \vdash m$ and $\nu \vdash h$.

Outline

Small subgroups

Multiplicity-free characters of S_n have degree at most $a_n := \sum_{\lambda \vdash n} \chi^\lambda(1)$.

Small subgroups

Multiplicity-free characters of S_n have degree at most $a_n := \sum_{\lambda \vdash n} \chi^{\lambda}(1)$. Consequently, multiplicity-free subgroups G satisfy $|G| \geq n!/a_n$.

Small subgroups

Multiplicity-free characters of S_n have degree at most $a_n := \sum_{\lambda \vdash n} \chi^{\lambda}(1)$.
Consequently, multiplicity-free subgroups G satisfy $|G| \geq n!/a_n$.
In fact a_n counts the involutions of S_n and thus for $n \geq 7$,
$a_n < n!/\left(2\lceil \frac{n}{2} \rceil!\right)$

Small subgroups

Multiplicity-free characters of S_n have degree at most $a_n := \sum_{\lambda \vdash n} \chi^\lambda(1)$.
Consequently, multiplicity-free subgroups G satisfy $|G| \geq n!/a_n$.
In fact a_n counts the involutions of S_n and thus for $n \geq 7$,
$a_n < n!/\left(2\lceil \frac{n}{2} \rceil !\right)$ (and for $n \geq 11$, also $a_n < n!/(2^{n-1})$).

Small subgroups

Multiplicity-free characters of S_n have degree at most $a_n := \sum_{\lambda \vdash n} \chi^\lambda(1)$.
Consequently, multiplicity-free subgroups G satisfy $|G| \geq n!/a_n$.
In fact a_n counts the involutions of S_n and thus for $n \geq 7$,
$a_n < n!/\left(2\lceil \frac{n}{2} \rceil !\right)$ (and for $n \geq 11$, also $a_n < n!/(2^{n-1})$).

Theorem (Maróti, 2002)

Let G be a primitive subgroup of S_n which is not S_n or A_n. Then the order of G is less or equal to 2^{n-1} or $n \leq 24$.

Small subgroups

Multiplicity-free characters of S_n have degree at most $a_n := \sum_{\lambda \vdash n} \chi^\lambda(1)$.
Consequently, multiplicity-free subgroups G satisfy $|G| \geq n!/a_n$.
In fact a_n counts the involutions of S_n and thus for $n \geq 7$,
$a_n < n!/\left(2\lceil \frac{n}{2} \rceil !\right)$ (and for $n \geq 11$, also $a_n < n!/(2^{n-1})$).

Theorem (Maróti, 2002)

Let G be a primitive subgroup of S_n which is not S_n or A_n. Then the order of G is less or equal to 2^{n-1} or $n \leq 24$.

The lower bound on $|G|$ also eliminates, for example, the diagonal subgroup of $S_k \times S_k$.

Non-transitive subgroups

Multiplicity-free subgroups have at most two orbits as shown by Stembridge.

Theorem (Stembridge, 2001)

Let $n = a + b + c$ for positive integers a, b and c. Then $S_a \times S_b \times S_c$ is not multiplicity-free.

Non-transitive subgroups

Multiplicity-free subgroups have at most two orbits as shown by Stembridge.

Theorem (Stembridge, 2001)

Let $n = a + b + c$ for positive integers a, b and c. Then $S_a \times S_b \times S_c$ is not multiplicity-free.

With an extra work I have shown that multiplicity-free subgroups $K \times L \leq S_k \times S_l$ satisfy either $K = S_k$ or $K = A_k$ (or $L = S_l$ or $L = A_l$).

Non-transitive subgroups

Multiplicity-free subgroups have at most two orbits as shown by Stembridge.

Theorem (Stembridge, 2001)

Let $n = a + b + c$ for positive integers a, b and c. Then $S_a \times S_b \times S_c$ is not multiplicity-free.

With an extra work I have shown that multiplicity-free subgroups $K \times L \leq S_k \times S_l$ satisfy either $K = S_k$ or $K = A_k$ (or $L = S_l$ or $L = A_l$). Moreover if $K = A_k$, then $l \leq 9$.

Non-transitive subgroups

Multiplicity-free subgroups have at most two orbits as shown by Stembridge.

Theorem (Stembridge, 2001)

Let $n = a + b + c$ for positive integers a, b and c. Then $S_a \times S_b \times S_c$ is not multiplicity-free.

With an extra work I have shown that multiplicity-free subgroups $K \times L \leq S_k \times S_l$ satisfy either $K = S_k$ or $K = A_k$ (or $L = S_l$ or $L = A_l$). Moreover if $K = A_k$, then $l \leq 9$.

This impacts non-elementary irreducible induced-multiplicity-free characters of wreath products.

Outline

Index two subgroups

Suppose that $G \neq N := G \cap A_n$.

Index two subgroups

Suppose that $G \neq N := G \cap A_n$. If ρ is an irreducible character of G and $\rho \times \mathrm{sgn} \neq \rho$, then $\kappa := \rho\big\downarrow_N^G$ is irreducible

Index two subgroups

Suppose that $G \neq N := G \cap A_n$. If ρ is an irreducible character of G and $\rho \times \mathrm{sgn} \neq \rho$, then $\kappa := \rho\downarrow_N^G$ is irreducible and $\kappa\uparrow_N^{S_n} = \rho\uparrow_G^{S_n} + \rho\uparrow_G^{S_n} \times \mathrm{sgn}$.

Index two subgroups

Suppose that $G \neq N := G \cap A_n$. If ρ is an irreducible character of G and $\rho \times \text{sgn} \neq \rho$, then $\kappa := \rho\downarrow_N^G$ is irreducible and $\kappa\uparrow_N^{S_n} = \rho\uparrow_G^{S_n} + \rho\uparrow_G^{S_n} \times \text{sgn}$. Thus κ is induced-multiplicity-free if and only if

- ρ is induced-multiplicity-free

Index two subgroups

Suppose that $G \neq N := G \cap A_n$. If ρ is an irreducible character of G and $\rho \times \mathrm{sgn} \neq \rho$, then $\kappa := \rho\downarrow_N^G$ is irreducible and $\kappa\uparrow_N^{S_n} = \rho\uparrow_G^{S_n} + \rho\uparrow_G^{S_n} \times \mathrm{sgn}$. Thus κ is induced-multiplicity-free if and only if

- ρ is induced-multiplicity-free, and
- $\rho\uparrow_G^{S_n}$ does not have constituents of the form χ^λ and $\chi^{\lambda'}$ for some λ.

Index two subgroups

Suppose that $G \neq N := G \cap A_n$. If ρ is an irreducible character of G and $\rho \times \text{sgn} \neq \rho$, then $\kappa := \rho\big\downarrow_N^G$ is irreducible and $\kappa\big\uparrow_N^{S_n} = \rho\big\uparrow_G^{S_n} + \rho\big\uparrow_G^{S_n} \times \text{sgn}$. Thus κ is induced-multiplicity-free if and only if

- ρ is induced-multiplicity-free, and
- $\rho\big\uparrow_G^{S_n}$ does not have constituents of the form χ^λ and $\chi^{\lambda'}$ for some λ.

Example: Let $G = S_2 \times S_3$ and $\rho = \chi^{(2)} \boxtimes \chi^{(2,1)}$. Since $\rho\big\uparrow_G^{S_5}$ decomposes as $\chi^{(4,1)} + \chi^{(3,2)} + \chi^{(3,1^2)} + \chi^{(2^2,1)}$, the character $\rho\big\downarrow_N^G$ is not induced-multiplicity-free.

Index two subgroups

Suppose that $G \neq N := G \cap A_n$. If ρ is an irreducible character of G and $\rho \times \mathrm{sgn} \neq \rho$, then $\kappa := \rho\big\downarrow_N^G$ is irreducible and $\kappa\big\uparrow_N^{S_n} = \rho\big\uparrow_G^{S_n} + \rho\big\uparrow_G^{S_n} \times \mathrm{sgn}$. Thus κ is induced-multiplicity-free if and only if

- ρ is induced-multiplicity-free, and
- $\rho\big\uparrow_G^{S_n}$ does not have constituents of the form χ^λ and $\chi^{\lambda'}$ for some λ.

Example: Let $G = S_2 \times S_3$ and $\rho = \chi^{(2)} \boxtimes \chi^{(2,1)}$. Since $\rho\big\uparrow_G^{S_5}$ decomposes as $\chi^{(4,1)} + \chi^{(3,2)} + \chi^{(3,1^2)} + \chi^{(2^2,1)}$, the character $\rho\big\downarrow_N^G$ is not induced-multiplicity-free.

Formula for $\left(\chi^{(a^b)} \wr \chi^{\nu}\right) \uparrow_{S_{ab} \wr S_2}^{S_{2ab}}$

For $a, b \in \mathbb{N}$ we say that a partition λ of size $2ab$ is (a, b)-*birectangular* if for all $1 \leq i \leq 2b$ the equality $\lambda_i + \lambda_{2b+1-i} = 2a$ holds true.

Formula for $\left(\chi^{(a^b)} \wr \chi^{\nu}\right) \uparrow_{S_{ab} \wr S_2}^{S_{2ab}}$

For $a, b \in \mathbb{N}$ we say that a partition λ of size $2ab$ is (a, b)-*birectangular* if for all $1 \leq i \leq 2b$ the equality $\lambda_i + \lambda_{2b+1-i} = 2a$ holds true.

Proposition

Let $\rho = \chi^{(a^b)} \wr \chi^{\nu}$. The character $\rho\uparrow_{S_{ab} \wr S_2}^{S_{2ab}}$ with $\nu = (2)$ and $\nu = (1^2)$ is multiplicity-free with constituents labelled by (a, b)-birectangular partitions λ such that $\lambda_1 + \cdots + \lambda_b$ is even, respectively, odd.

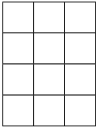
Formula for $\left(\chi^{(a^b)} \wr \chi^{\nu}\right)\uparrow_{S_{ab}\wr S_2}^{S_{2ab}}$

For $a, b \in \mathbb{N}$ we say that a partition λ of size $2ab$ is (a, b)-*birectangular* if for all $1 \leq i \leq 2b$ the equality $\lambda_i + \lambda_{2b+1-i} = 2a$ holds true.

Proposition

Let $\rho = \chi^{(a^b)} \wr \chi^{\nu}$. The character $\rho\uparrow_{S_{ab}\wr S_2}^{S_{2ab}}$ with $\nu = (2)$ and $\nu = (1^2)$ is multiplicity-free with constituents labelled by (a, b)-birectangular partitions λ such that $\lambda_1 + \cdots + \lambda_b$ is even, respectively, odd.

Example: Let $\rho_1 = \chi^{(3^2)} \wr \chi^{(2)}$ and $\rho_2 = \chi^{(3^2)} \wr \chi^{(1^2)}$. One of the irreducible constituents of $\rho_1\uparrow_{S_6\wr S_2}^{S_{12}}$ is labelled by (3^4).



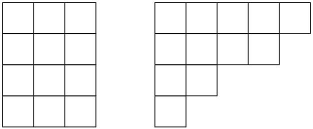
Formula for $\left(\chi^{(a^b)} \wr \chi^{\nu}\right)\uparrow_{S_{ab}\wr S_2}^{S_{2ab}}$

For $a, b \in \mathbb{N}$ we say that a partition λ of size $2ab$ is (a,b)-*birectangular* if for all $1 \leq i \leq 2b$ the equality $\lambda_i + \lambda_{2b+1-i} = 2a$ holds true.

Proposition

Let $\rho = \chi^{(a^b)} \wr \chi^{\nu}$. The character $\rho\uparrow_{S_{ab}\wr S_2}^{S_{2ab}}$ with $\nu = (2)$ and $\nu = (1^2)$ is multiplicity-free with constituents labelled by (a,b)-birectangular partitions λ such that $\lambda_1 + \cdots + \lambda_b$ is even, respectively, odd.

Example: Let $\rho_1 = \chi^{(3^2)} \wr \chi^{(2)}$ and $\rho_2 = \chi^{(3^2)} \wr \chi^{(1^2)}$. One of the irreducible constituents of $\rho_1\uparrow_{S_6\wr S_2}^{S_{12}}$ is labelled by (3^4). If we consider $\rho_2\uparrow_{S_6\wr S_2}^{S_{12}}$ instead, we obtain a label $(5,4,2,1)$.



Birectangular partitions

Proposition (T, 2023)

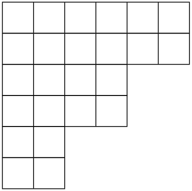
Let $a > b$ be positive integers and write $d = a - b$. There is a partition λ such that λ and λ' are (a, b)-birectangular if and only if $d|a$. Moreover, in such a case $\lambda = ((2b)^{2d}, (2b - 2d)^{2d}, \dots, (2d)^{2d})$.

Birectangular partitions

Proposition (T, 2023)

Let $a > b$ be positive integers and write $d = a - b$. There is a partition λ such that λ and λ' are (a, b)-birectangular if and only if $d|a$. Moreover, in such a case $\lambda = ((2b)^{2d}, (2b-2d)^{2d}, \dots, (2d)^{2d})$.

Example: The unique partition $\lambda = (6^2, 4^2, 2^2)$ such that λ and λ' are both $(4, 3)$-birectangular.



Example: birectangular partitions

Recall the setting $G \neq N := G \cap A_n$ and the following results.

Example: birectangular partitions

Recall the setting $G \neq N := G \cap A_n$ and the following results.

- If ρ is an irreducible induced-multiplicity-free character of G, then (under mild conditions) $\rho\downarrow_N^G$ is induced-multiplicity-free if and only if there are no constituents χ^λ and $\chi^{\lambda'}$ of $\rho\uparrow_G^{S_n}$.

Example: birectangular partitions

Recall the setting $G \neq N := G \cap A_n$ and the following results.

- If ρ is an irreducible induced-multiplicity-free character of G, then (under mild conditions) $\rho\big\downarrow_N^G$ is induced-multiplicity-free if and only if there are no constituents χ^λ and $\chi^{\lambda'}$ of $\rho\big\uparrow_G^{S_n}$.
- If $\rho = \chi^{(a^b)} \wr \chi^\nu$, then $\rho\big\uparrow_{S_{ab} \wr S_2}^{S_{2ab}}$ is multiplicity-free with irreducible constituents labelled by (a, b)-birectangular partitions λ such that $\lambda_1 + \cdots + \lambda_b$ is even for $\nu = (2)$, respectively, odd for $\nu = (1^2)$.

Example: birectangular partitions

Recall the setting $G \neq N := G \cap A_n$ and the following results.

- If ρ is an irreducible induced-multiplicity-free character of G, then (under mild conditions) $\rho\downarrow_N^G$ is induced-multiplicity-free if and only if there are no constituents χ^λ and $\chi^{\lambda'}$ of $\rho\uparrow_G^{S_n}$.
- If $\rho = \chi^{(a^b)} \wr \chi^\nu$, then $\rho\uparrow_{S_{ab}\wr S_2}^{S_{2ab}}$ is multiplicity-free with irreducible constituents labelled by (a, b)-birectangular partitions λ such that $\lambda_1 + \cdots + \lambda_b$ is even for $\nu = (2)$, respectively, odd for $\nu = (1^2)$.
- There is λ such that λ and λ' are (a, b)-birectangular if and only if $a - b \mid a$. In such a case λ is self-conjugate and even.

Example: birectangular partitions

Recall the setting $G \neq N := G \cap A_n$ and the following results.

- If ρ is an irreducible induced-multiplicity-free character of G, then (under mild conditions) $\rho\downarrow_N^G$ is induced-multiplicity-free if and only if there are no constituents χ^λ and $\chi^{\lambda'}$ of $\rho\uparrow_G^{S_n}$.
- If $\rho = \chi^{(a^b)} \wr \chi^\nu$, then $\rho\uparrow_{S_{ab}\wr S_2}^{S_{2ab}}$ is multiplicity-free with irreducible constituents labelled by (a, b)-birectangular partitions λ such that $\lambda_1 + \cdots + \lambda_b$ is even for $\nu = (2)$, respectively, odd for $\nu = (1^2)$.
- There is λ such that λ and λ' are (a, b)-birectangular if and only if $a - b \mid a$. In such a case λ is self-conjugate and even.

Example: Let $G = S_{15} \wr S_2$ and $\rho = \chi^{(5^3)} \wr \chi^{(2)}$. Then $\rho\downarrow_N^G$ is induced-multiplicity-free.

Example: birectangular partitions

Recall the setting $G \neq N := G \cap A_n$ and the following results.

- If ρ is an irreducible induced-multiplicity-free character of G, then (under mild conditions) $\rho\downarrow_N^G$ is induced-multiplicity-free if and only if there are no constituents χ^λ and $\chi^{\lambda'}$ of $\rho\uparrow_G^{S_n}$.
- If $\rho = \chi^{(a^b)} \wr \chi^\nu$, then $\rho\uparrow_{S_{ab}\wr S_2}^{S_{2ab}}$ is multiplicity-free with irreducible constituents labelled by (a, b)-birectangular partitions λ such that $\lambda_1 + \cdots + \lambda_b$ is even for $\nu = (2)$, respectively, odd for $\nu = (1^2)$.
- There is λ such that λ and λ' are (a, b)-birectangular if and only if $a - b \mid a$. In such a case λ is self-conjugate and even.

Example: Let $G = S_{15} \wr S_2$ and $\rho = \chi^{(5^3)} \wr \chi^{(1^2)}$. Then $\rho\downarrow_N^G$ is induced-multiplicity-free.

Example: birectangular partitions

Recall the setting $G \neq N := G \cap A_n$ and the following results.

- If ρ is an irreducible induced-multiplicity-free character of G, then (under mild conditions) $\rho\downarrow_N^G$ is induced-multiplicity-free if and only if there are no constituents χ^λ and $\chi^{\lambda'}$ of $\rho\uparrow_G^{S_n}$.
- If $\rho = \chi^{(a^b)} \wr \chi^\nu$, then $\rho\uparrow_{S_{ab}\wr S_2}^{S_{2ab}}$ is multiplicity-free with irreducible constituents labelled by (a, b)-birectangular partitions λ such that $\lambda_1 + \cdots + \lambda_b$ is even for $\nu = (2)$, respectively, odd for $\nu = (1^2)$.
- There is λ such that λ and λ' are (a, b)-birectangular if and only if $a - b \mid a$. In such a case λ is self-conjugate and even.

Example: Let $G = S_{12} \wr S_2$ and $\rho = \chi^{(4^3)} \wr \chi^{(1^2)}$. Then $\rho\downarrow_N^G$ is induced-multiplicity-free.

Example: birectangular partitions

Recall the setting $G \neq N := G \cap A_n$ and the following results.

- If ρ is an irreducible induced-multiplicity-free character of G, then (under mild conditions) $\rho\downarrow_N^G$ is induced-multiplicity-free if and only if there are no constituents χ^λ and $\chi^{\lambda'}$ of $\rho\uparrow_G^{S_n}$.
- If $\rho = \chi^{(a^b)} \wr \chi^\nu$, then $\rho\uparrow_{S_{ab}\wr S_2}^{S_{2ab}}$ is multiplicity-free with irreducible constituents labelled by (a, b)-birectangular partitions λ such that $\lambda_1 + \cdots + \lambda_b$ is even for $\nu = (2)$, respectively, odd for $\nu = (1^2)$.
- There is λ such that λ and λ' are (a, b)-birectangular if and only if $a - b \mid a$. In such a case λ is self-conjugate and even.

Example: Let $G = S_{12} \wr S_2$ and $\rho = \chi^{(4^3)} \wr \chi^{(2)}$. Then $\rho\downarrow_N^G$ is not induced-multiplicity-free.

Outline

Multiplicity-free subgroups of S_n (light version)

Theorem (T, 2023)

Suppose that $n \geq 66$. Multiplicity-free subgroups of S_n are given by certain subgroups of index $1, 2$ and 4 of

1. $S_k \times S_l$,
2. $S_k \times S_m \wr S_2$,
3. $S_k \times S_2 \wr S_h$,
4. $S_m \wr S_3$,
5. $S_k \times L$ where L is one of $\mathrm{P\Gamma L}_2(\mathbb{F}_8) \leq S_9$, $\mathrm{ASL}_3(\mathbb{F}_2) \leq S_8$, $\mathrm{PGL}_2(\mathbb{F}_5) \leq S_6$ and $\mathrm{AGL}_1(\mathbb{F}_5) \leq S_5$,

where $k, l \geq 0$, $m \geq 2$ and $h \geq 3$.

Multiplicity-free subgroups of S_n (light version)

Theorem (T, 2023)

Suppose that $n \geq 66$. Multiplicity-free subgroups of S_n are given by certain subgroups of index $1, 2$ and 4 of

1. $S_k \times S_l$,
2. $S_k \times S_m \wr S_2$,
3. $S_k \times S_2 \wr S_h$,
4. $S_m \wr S_3$,
5. $S_k \times L$ where L is one of $\mathrm{P\Gamma L}_2(\mathbb{F}_8) \leq S_9$, $\mathrm{ASL}_3(\mathbb{F}_2) \leq S_8$, $\mathrm{PGL}_2(\mathbb{F}_5) \leq S_6$ and $\mathrm{AGL}_1(\mathbb{F}_5) \leq S_5$,

where $k, l \geq 0$, $m \geq 2$ and $h \geq 3$.

The trivial characters of groups in (2), (3) and (4) are not induced-multiplicity-free, provided $k \geq 2$.

Multiplicity-free subgroups of S_n (full version)

Theorem (T, 2023)

Let $n \geq 66$. A subgroup $G \leq S_n$ is multiplicity-free if and only if it belongs to the list (throughout $k, l \geq 1$, $m \geq 2$ and $h \geq 3$):

1. S_n and A_n,
2. $S_k \times S_l$, $(S_k \times S_l) \cap A_{k+l}$ and $A_k \times S_l$ with $k \neq 2$ and $A_k \times A_l$ with $k, l \neq 2$,
3. $S_k \times S_m \wr S_2$ and for $k \notin \{2m-3, 2m-2, 2m-1, 2m\}$ also groups $(S_k \times S_m \wr S_2) \cap A_{k+2m}$ and $T_{k,m,2}$,
4. $S_k \times S_2 \wr S_h$ and for $k \geq 2h+2$ also $(S_k \times S_2 \wr S_h) \cap A_{k+2h}$,
5. $S_m \wr S_2, (S_m \wr S_2) \cap A_{2m}, A_m \wr S_2$ and $T_{m,2}$,
6. $S_2 \wr S_h$ and $(S_2 \wr S_h) \cap A_{2h}$,
7. $S_m \wr S_3, (S_m \wr S_3) \cap A_{3m}$ and $T_{m,3}$,
8. $S_k \times L, A_k \times L$ and $(S_k \times L) \cap A_n$ where L is one of $\mathrm{P\Gamma L}(2,8), \mathrm{ASL}(3,2)$, $\mathrm{PGL}(2,5)$ and $\mathrm{AGL}(1,5)$,
9. $A_k \times S_2 \wr S_2, N_k$ and $T_{k,2,h}$ with $h \in \{3,4\}$,
10. $A_m \wr S_2$ with $n = 2m+1$ provided m is a square.

Multiplicity-free subgroups of S_n (full version)

Theorem (T, 2023)

Let $n \geq 66$. A subgroup $G \leq S_n$ is multiplicity-free if and only if it belongs to the list (throughout $k, l \geq 1$, $m \geq 2$ and $h \geq 3$):

1. S_n and A_n,
2. $S_k \times S_l$, $(S_k \times S_l) \cap A_{k+l}$ and $A_k \times S_l$ with $k \neq 2$ and $A_k \times A_l$ with $k, l \neq 2$,
3. $S_k \times S_m \wr S_2$ and for $k \notin \{2m-3, 2m-2, 2m-1, 2m\}$ also groups $(S_k \times S_m \wr S_2) \cap A_{k+2m}$ and $T_{k,m,2}$,
4. $S_k \times S_2 \wr S_h$ and for $k \geq 2h+2$ also $(S_k \times S_2 \wr S_h) \cap A_{k+2h}$,
5. $S_m \wr S_2, (S_m \wr S_2) \cap A_{2m}, A_m \wr S_2$ and $T_{m,2}$,
6. $S_2 \wr S_h$ and $(S_2 \wr S_h) \cap A_{2h}$,
7. $S_m \wr S_3, (S_m \wr S_3) \cap A_{3m}$ and $T_{m,3}$,
8. $S_k \times L, A_k \times L$ and $(S_k \times L) \cap A_n$ where L is one of $\mathrm{P\Gamma L}(2,8)$, $\mathrm{ASL}(3,2)$, $\mathrm{PGL}(2,5)$ and $\mathrm{AGL}(1,5)$,
9. $A_k \times S_2 \wr S_2, N_k$ and $T_{k,2,h}$ with $h \in \{3,4\}$,
10. $A_m \wr S_2$ with $n = 2m+1$ provided m is a square.

Induced-multiplicity-free characters of $(S_m \wr S_2) \cap A_{2m}$

Theorem (T, 2023)

Let $m \geq 37$. The elementary irreducible induced-multiplicity-free characters of $G = (S_m \wr S_2) \cap A_{2m}$ are the irreducible constituents of $(\chi^\mu \wr \chi^\nu) \downarrow_G^{S_m \wr S_2}$ with:

1. μ of the form (a^b) such that $a - b \nmid a$ or $\nu = (1^2)$;
2. μ of the form $(a+1, b)$ with $b > a+1$;
3. μ of the form $(a+1, a^{b-1}), (a^b, 1)$ or $(a^{b-1}, a-1)$ with $a > b+2$ and $a - b \nmid a$;
4. μ of the form $((2b)^{b-1}, 2b-1), (2b+1, (2b)^{b-1})$ or $(3b+1, (3b)^{2b-1})$;
5. μ of the form $(a^{a-1}, a-1)$ provided m is even.

Induced-multiplicity-free characters of $(S_m \wr S_2) \cap A_{2m}$

Theorem (T, 2023)

Let $m \geq 37$. The elementary irreducible induced-multiplicity-free characters of $G = (S_m \wr S_2) \cap A_{2m}$ are the irreducible constituents of $(\chi^\mu \wr \chi^\nu) \downarrow_G^{S_m \wr S_2}$ with:

1. μ of the form (a^b) such that $a - b \nmid a$ or $\nu = (1^2)$;
2. μ of the form $(a+1, b)$ with $b > a+1$;
3. μ of the form $(a+1, a^{b-1}), (a^b, 1)$ or $(a^{b-1}, a-1)$ with $a > b+2$ and $a - b \nmid a$;
4. μ of the form $((2b)^{b-1}, 2b-1), (2b+1, (2b)^{b-1})$ or $(3b+1, (3b)^{2b-1})$;
5. μ of the form $(a^{a-1}, a-1)$ provided m is even.

Induced-multiplicity-free characters of $(S_m \wr S_2) \cap A_{2m}$

Theorem (T, 2023)

Let $m \geq 37$. The elementary irreducible induced-multiplicity-free characters of $G = (S_m \wr S_2) \cap A_{2m}$ are the irreducible constituents of $(\chi^\mu \wr \chi^\nu) \downarrow_G^{S_m \wr S_2}$ with:

1. μ of the form (a^b) such that $a - b \nmid a$ or $\nu = (1^2)$;
2. μ of the form $(a+1, b)$ with $b > a+1$;
3. μ of the form $(a+1, a^{b-1}), (a^b, 1)$ or $(a^{b-1}, a-1)$ with $a > b+2$ and $a - b \nmid a$;
4. μ of the form $((2b)^{b-1}, 2b-1), (2b+1, (2b)^{b-1})$ or $(3b+1, (3b)^{2b-1})$;
5. μ of the form $(a^{a-1}, a-1)$ provided m is even.