Infinite friezes in affine type D

jt with K. Baur, E. Gunawan, G. Todorov and E. Yildirim

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Groups and their actions: algebraic, geometric and combinatorial aspects

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0) Introduction: Conway - Coxeter Friezes

Def: Staggered array of possibly infinitely many rows of integers, starting with a row of 1's, satisfying the diamond rule: for each diamond
$$\begin{array}{ccc} & b & \\ a & & d \\ & c & \end{array}$$
we have: $\boxed{ad - bc = 1}$.

Example:

$$\begin{array}{ccccccccccccccccccccc}
\cdots & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & \cdots \\
\cdots & & 2 & & 2 & & 2 & & 1 & & 4 & & 1 & & 2 & & 2 & & 2 & & \cdots \\
\cdots & 1 & & 3 & & 3 & & 1 & & 3 & & 3 & & 1 & & 3 & & 3 & & 1 & \cdots \\
\cdots & & 1 & & 4 & & 1 & & 2 & & 2 & & 2 & & 1 & & 4 & & 1 & & \cdots \\
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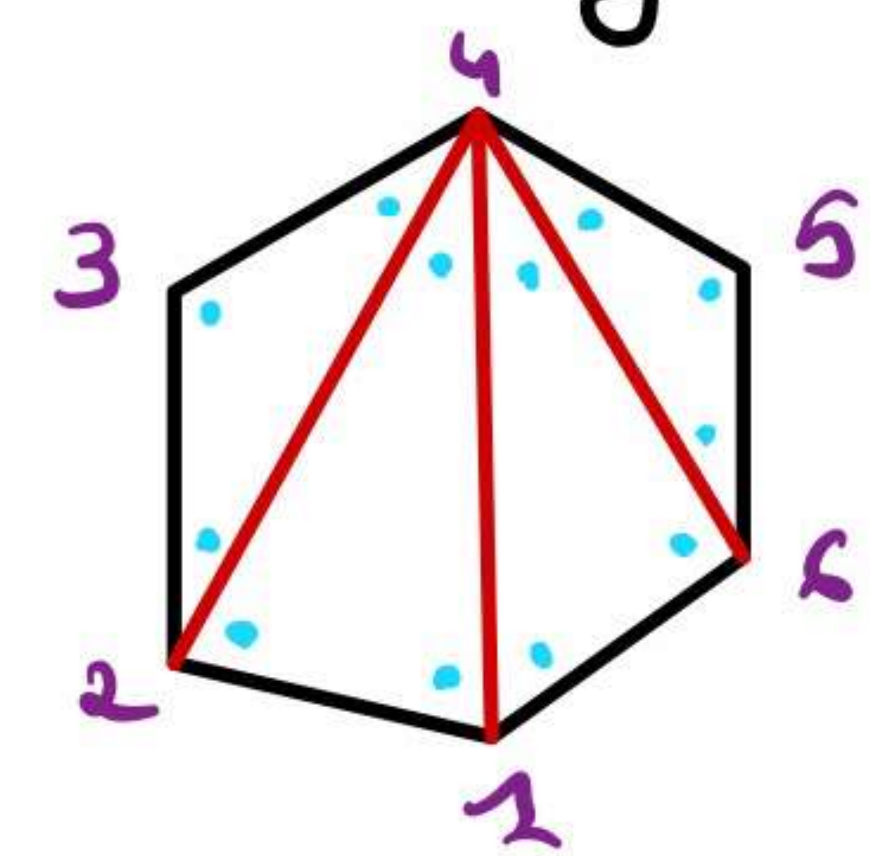

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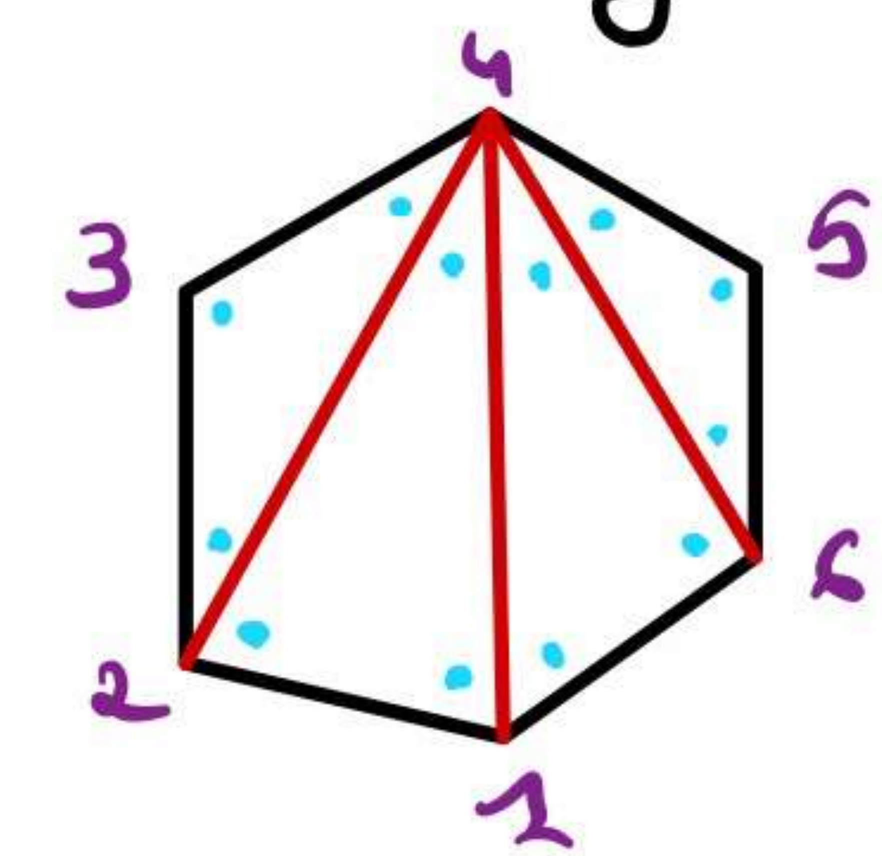

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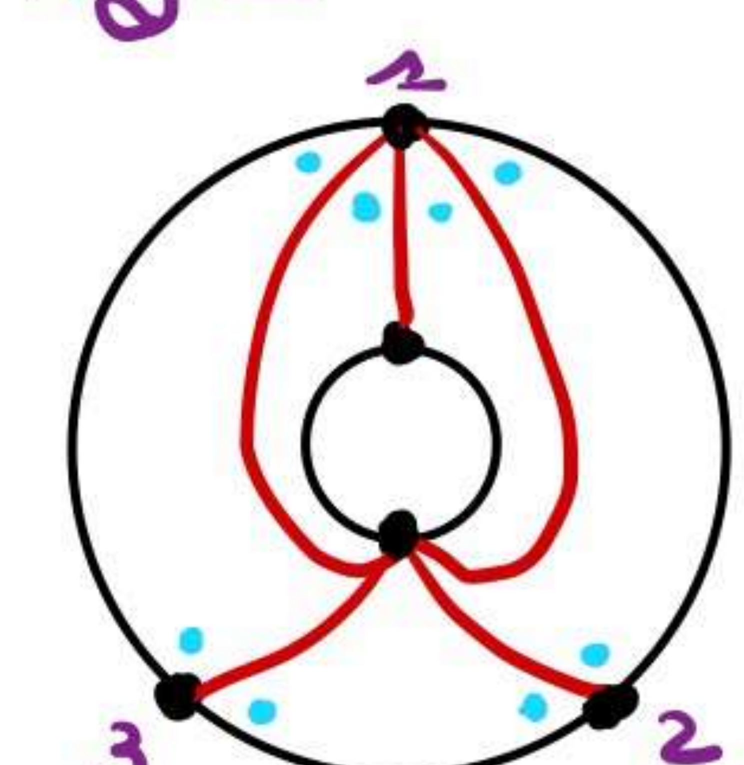

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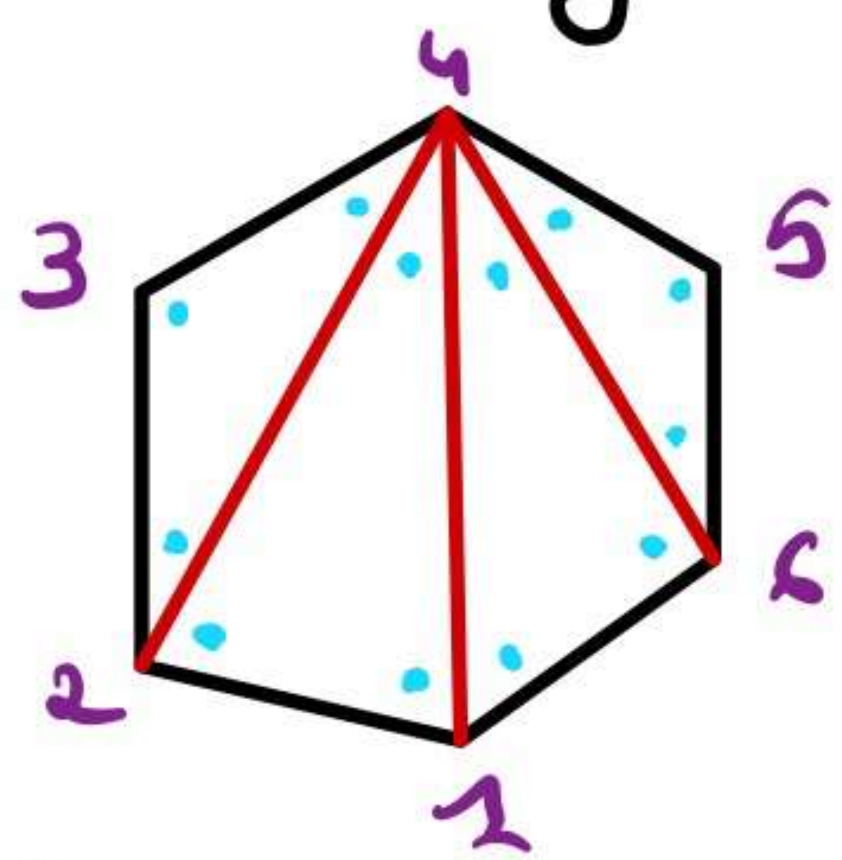

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=: quiddity sequence

$$
\begin{array}{ccccccccccccccccc}
\cdots & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & \cdots \\
\cdots & & \boxed{4 \quad 2 \quad 2} & & & & 4 & & 2 & & 2 & & 4 & & & & \cdots \\
\cdots & 7 & & 7 & & 3 & & 7 & & 7 & & 3 & & 7 & & 7 & \cdots \\
\cdots & & 12 & & 10 & & 10 & & 12 & & 10 & & 10 & & 12 & & \cdots \\
\cdots & 17 & & 17 & & 33 & & 17 & & 17 & & 33 & & 17 & & 17 & \cdots \\
\cdots & & 24 & & 56 & & 56 & & 24 & & 56 & & 56 & & 24 & & \cdots \\
 & & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & &
\end{array}
$$

$$\frac{10 \times 12 - 1}{7} = 17$$

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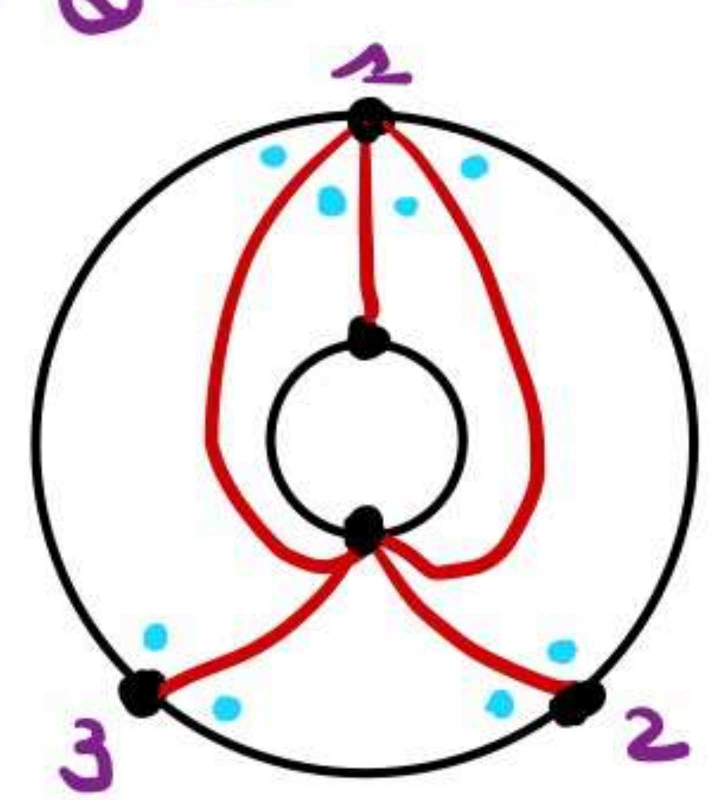

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$$\begin{array}{ccccccccccccccccc}
\cdots & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & \cdots \\
\cdots & & \boxed{4 & & 2 & & 2} & & 4 & & 2 & & 2 & & 4 & & \cdots \\
\cdots & 7 & & 7 & & 3 & & 7 & & 7 & & 3 & & 7 & & 7 & \cdots \\
\cdots & & 12 & & 10 & & 10 & & 12 & & 10 & & 10 & & 12 & & \cdots \\
\cdots & 17 & & 17 & & 33 & & 17 & & 17 & & 33 & & 17 & & 17 & \cdots \\
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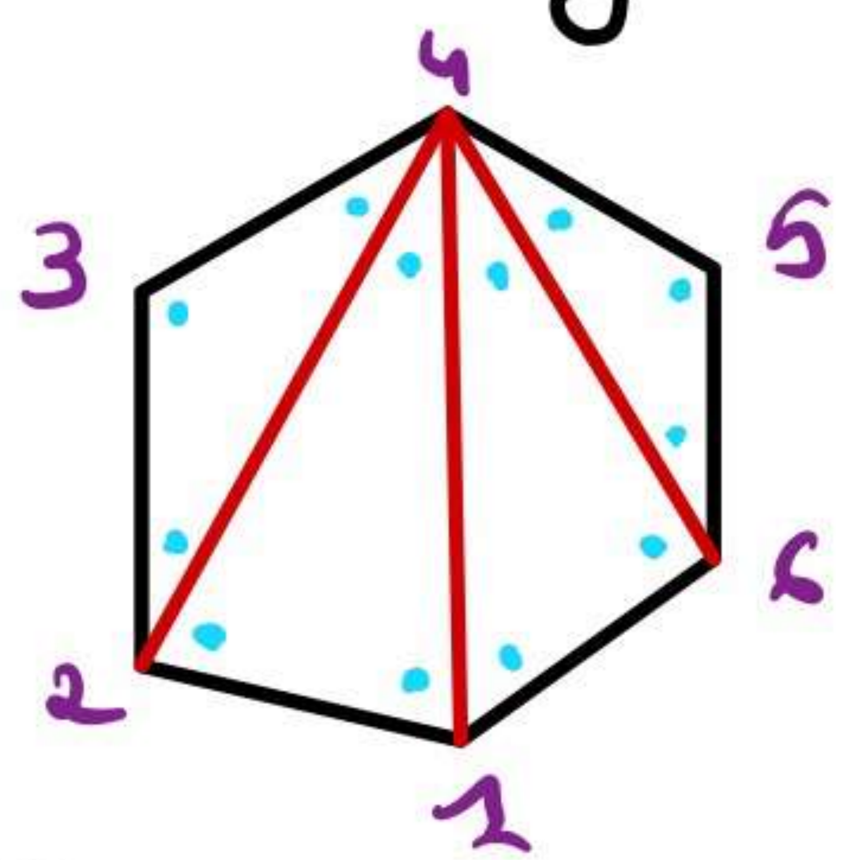

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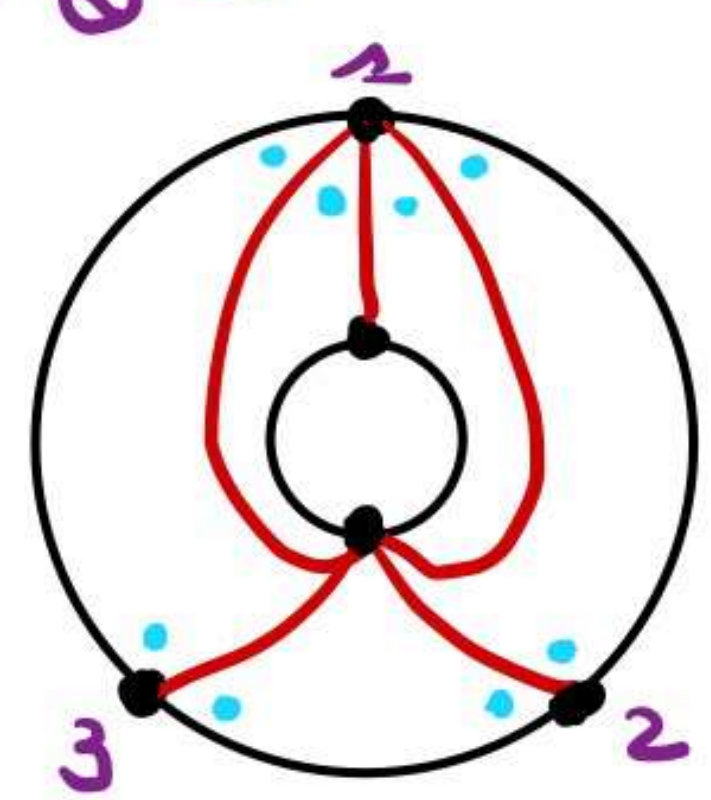

- [Baur-Parsons-Tschabold, 16]

$$\left\{ \begin{array}{c} n\text{-periodic} \\ \text{infinite friezes} \end{array} \right\} \overset{1:1}{\longleftrightarrow} \left\{ \begin{array}{c} \text{triangulations of annuli with } n \\ \text{marked points on the outer boundary} \end{array} \right\}$$

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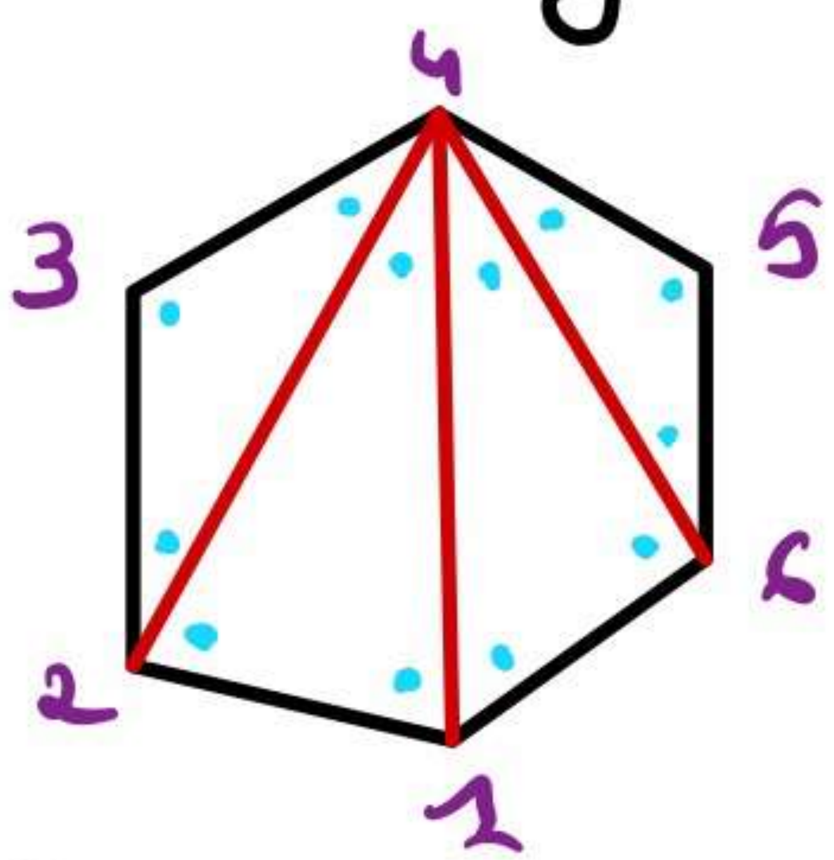

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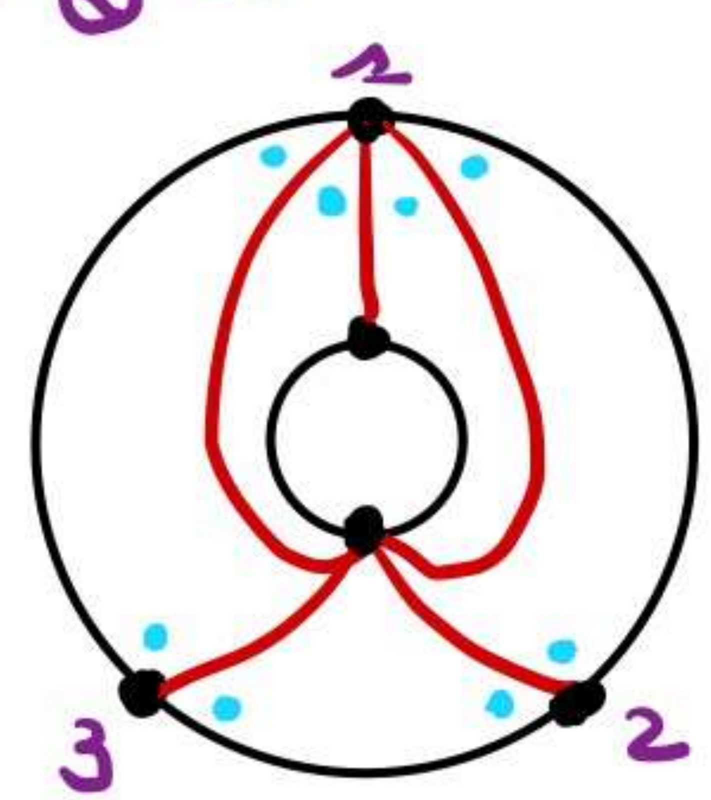

Def/prop: [B-Fellner-PT, 19] For any n-periodic frieze, the difference between the entry in row n and $n-2$ is constant: growth coefficient.

$$
\begin{array}{ccccccccccc}
 & & \multicolumn{3}{c}{=: \text{quiddity sequence}} & & & & & & \\
\cdots & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \cdots & \\
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Example: For the previous frieze the growth coefficient is
$s = 12 - 4 = 10 - 2 = \boxed{8}$.

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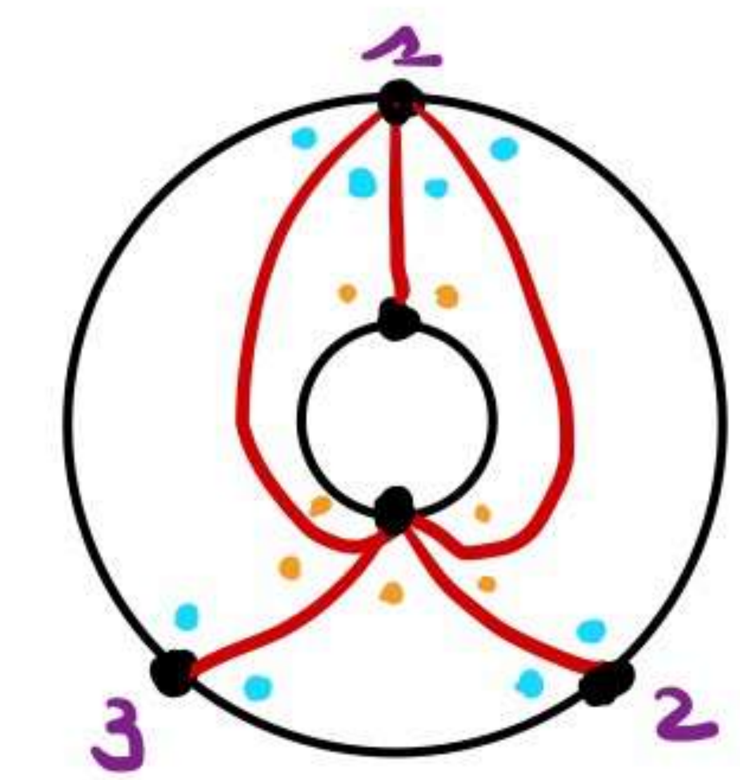

quiddity sequence for the inner boundary: $(2, 5)$

$$\begin{array}{ccccccc}
1 & & 1 & & 1 & & 1 \\
& \boxed{2 \quad 5} & & & & 2 & \\
9 & & 9 & & 9 & & 9
\end{array}$$

$$s = 9 - 1 = 9$$

Rk Not true for a triangulation of a pair of pants.

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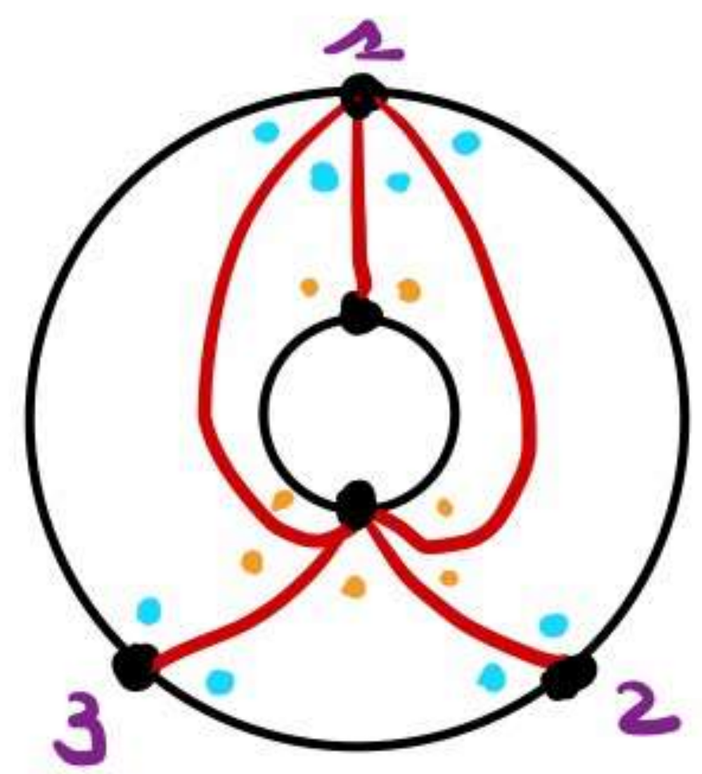

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Let S be a connected, oriented, Riemann surface with boundary. Let M be a finite set of marked points on the boundary of S, or in the interior (punctures).

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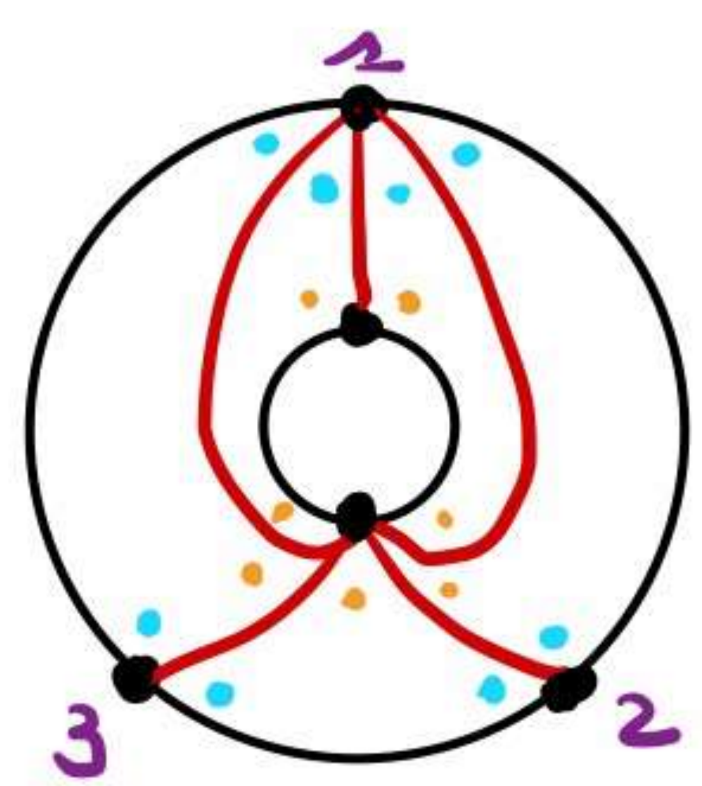

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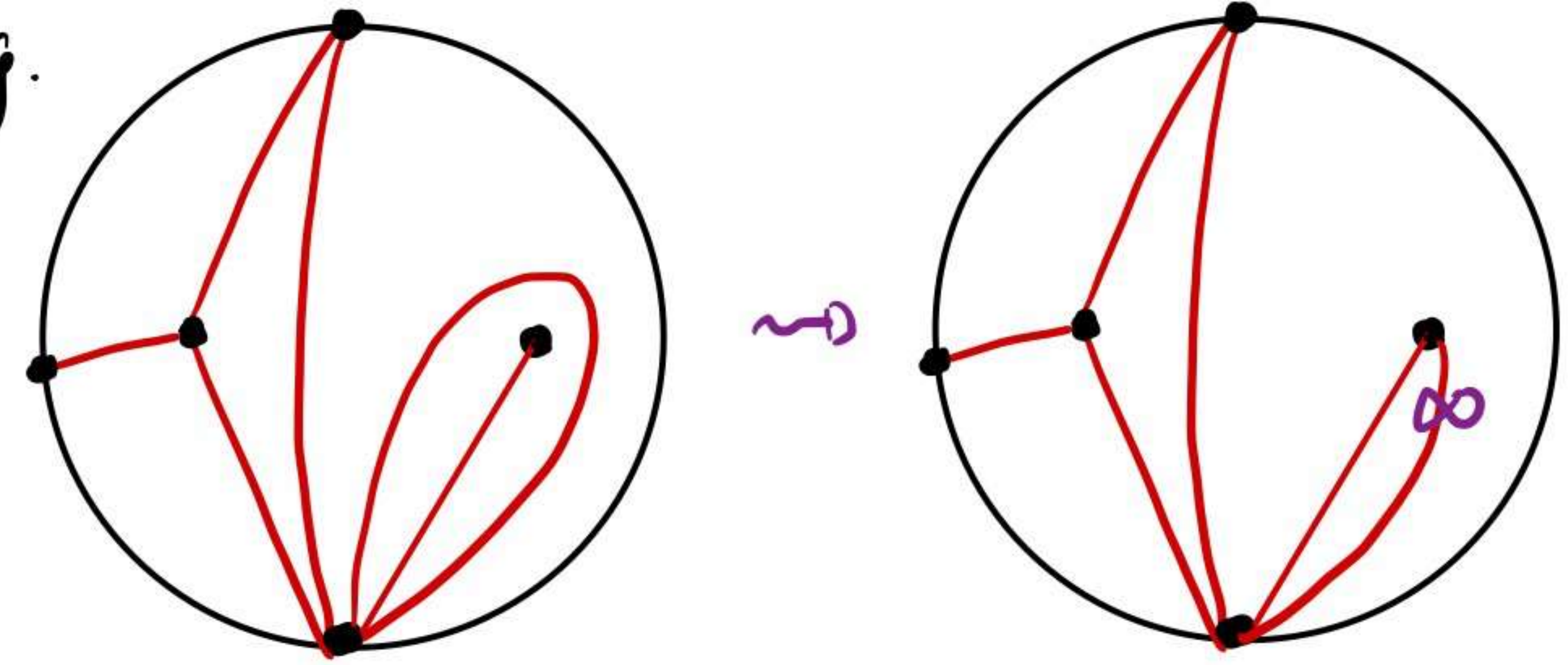
Let S be a connected, oriented, Riemann surface with boundary. Let M be a finite set of marked points on the boundary of S, or in the interior (punctures).

Def: A tagged arc is a curve with endpoints in M, which does not intersect itself or the boundary (except on its endpoints), does not cut out an unpunctured monogon or is an edge of an unpunctured digon.

Each end is tagged: either notched or unotched.

An endpoint on the boundary is always unotched and a loop has the same tagging at both ends.

e.g.



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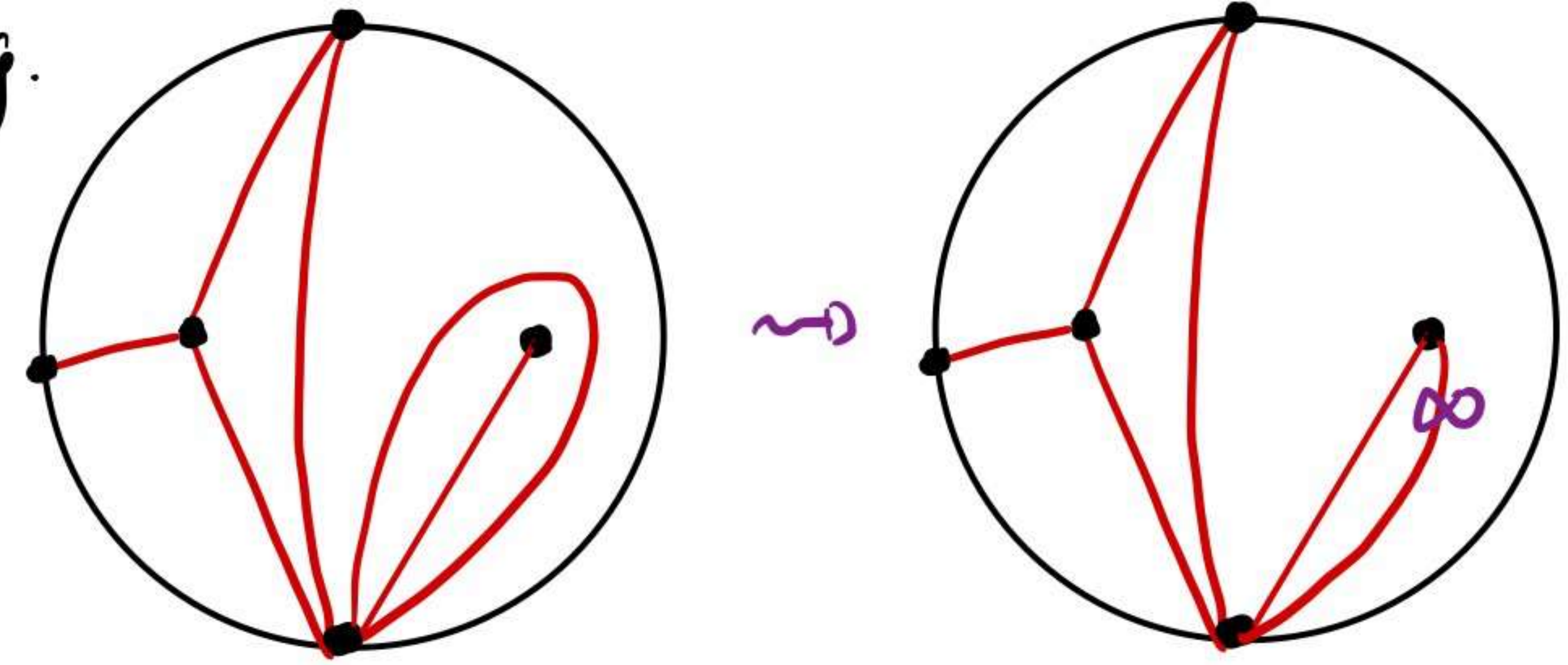

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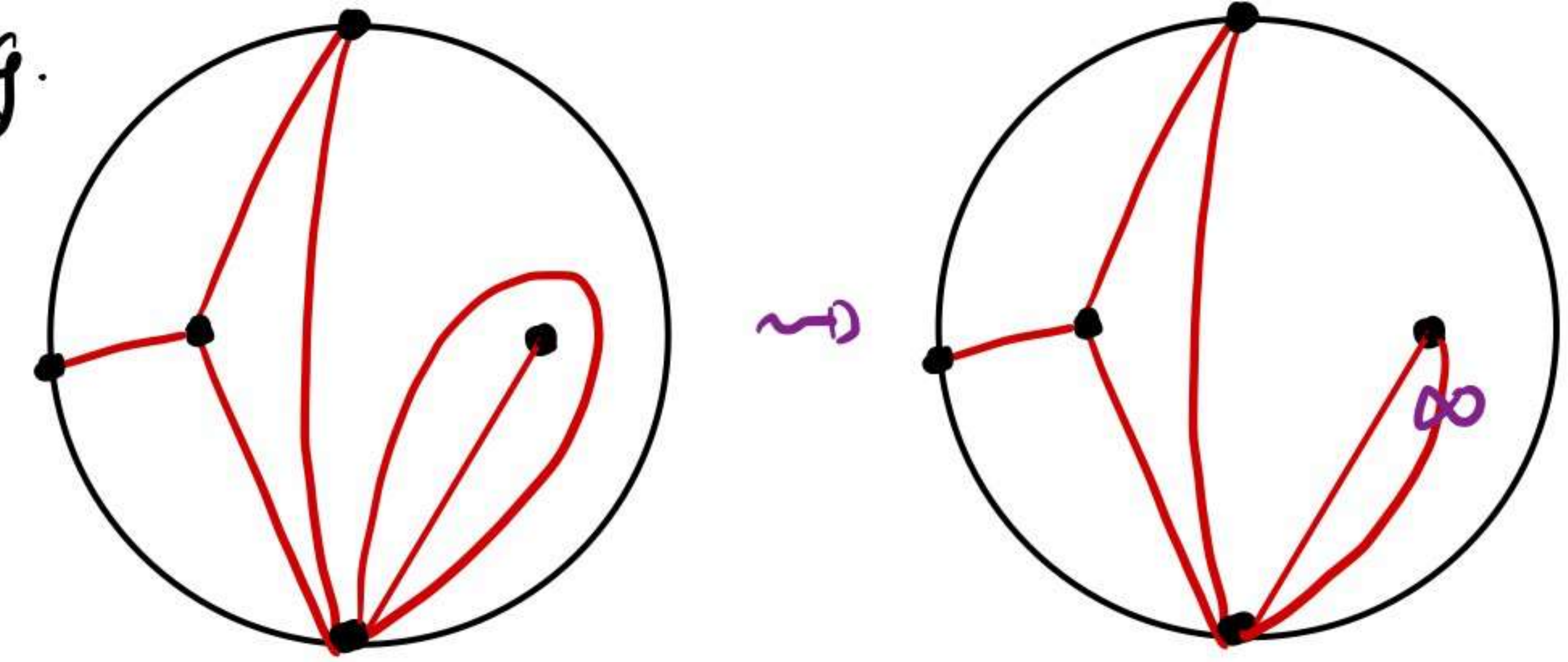

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$Q_T = (Q_0, Q_1)$, $Q_0 = \{\text{arcs in } T\}$

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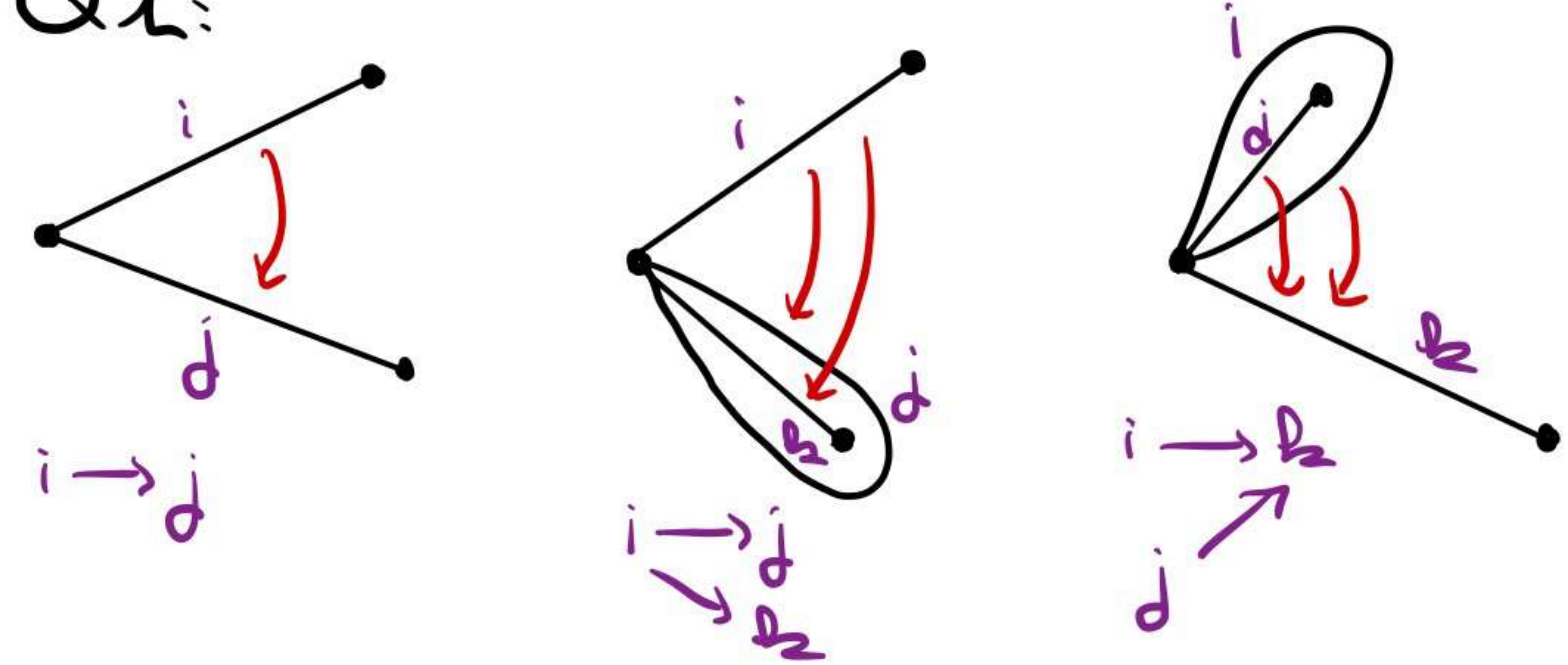

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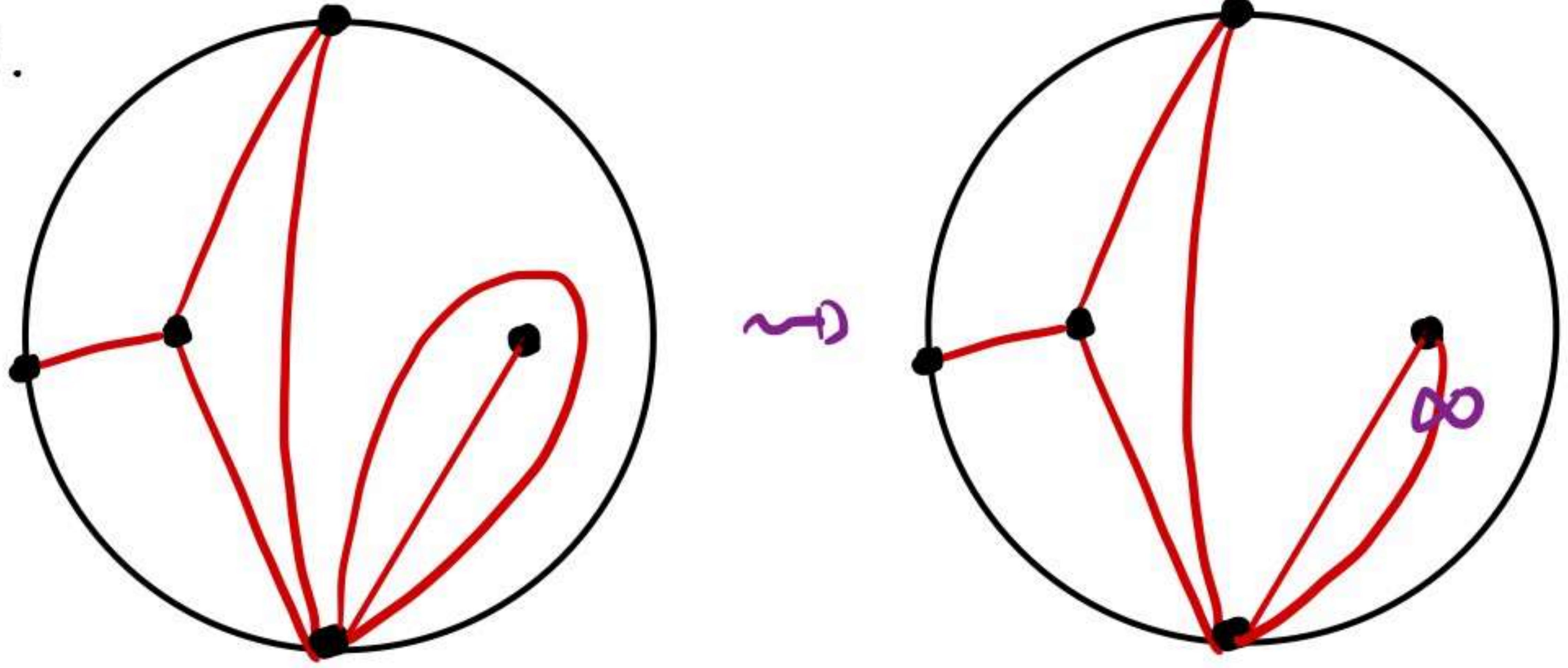

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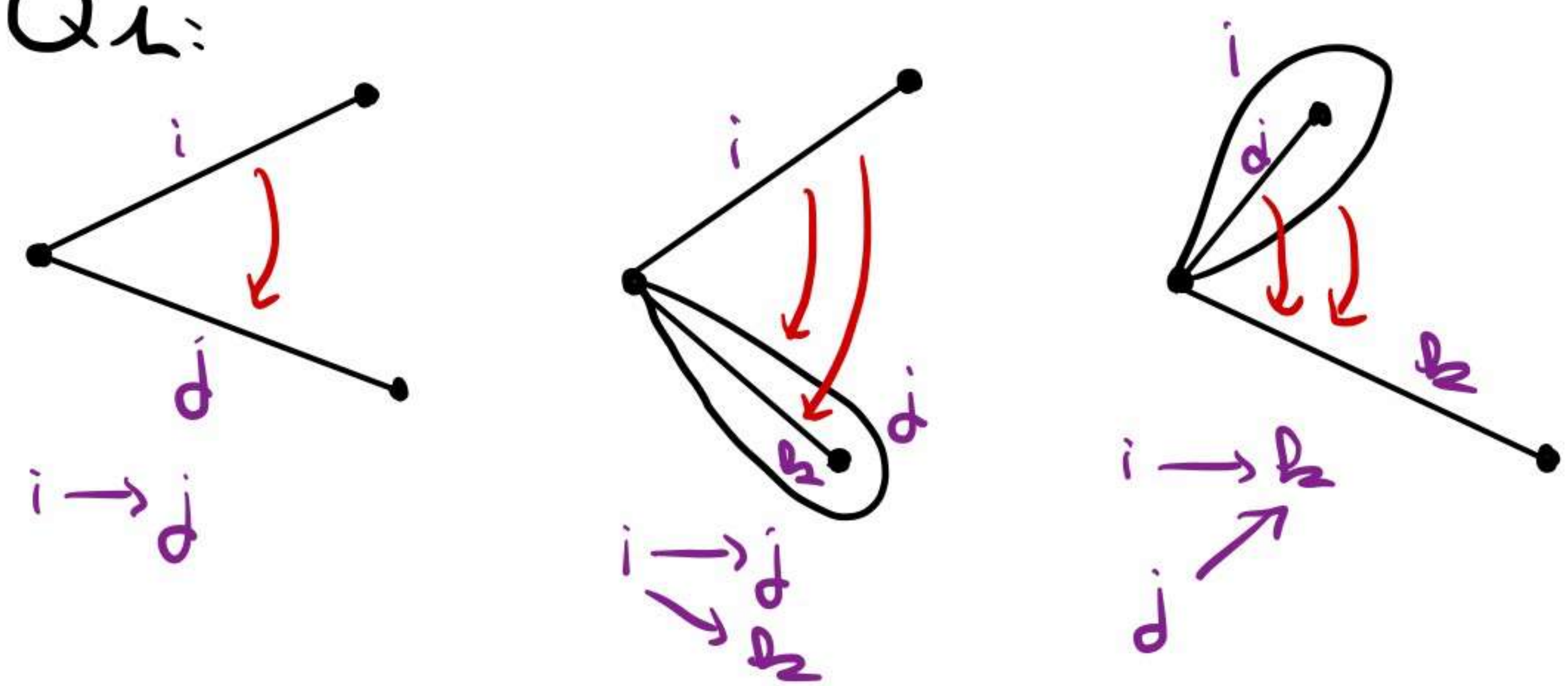

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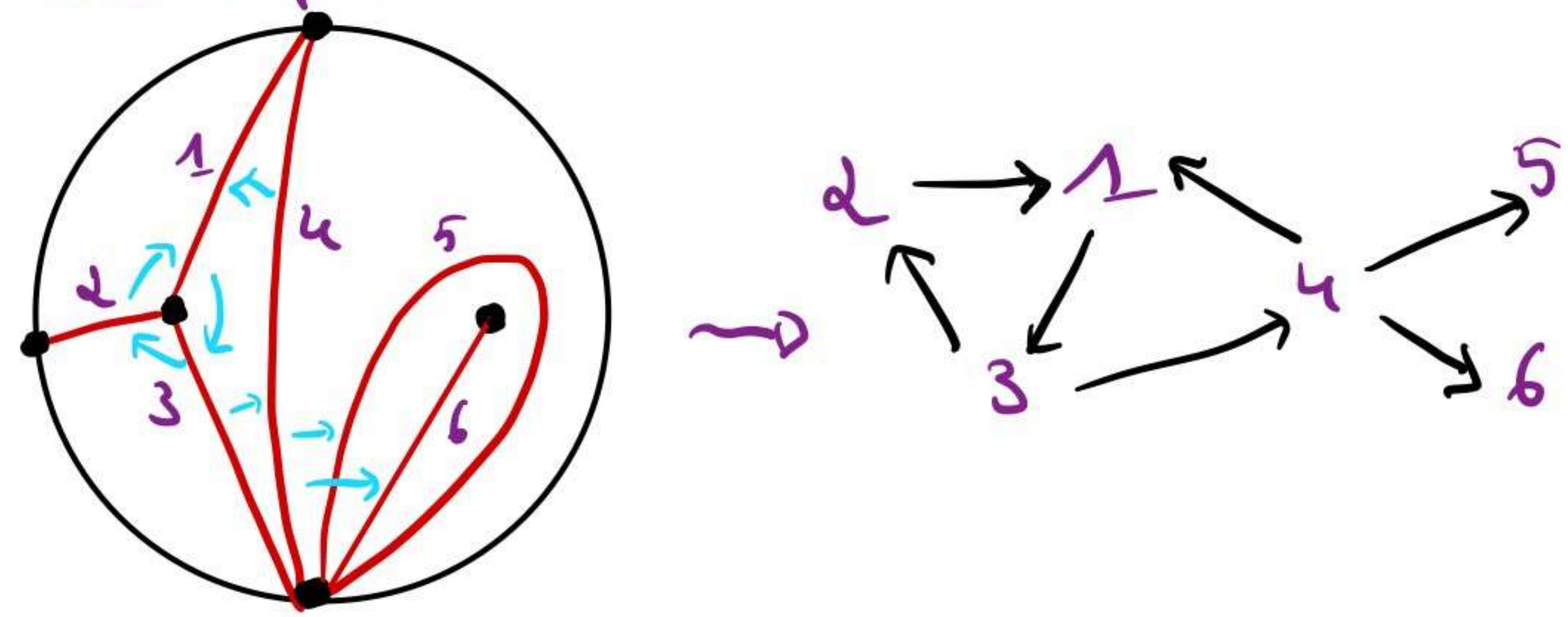

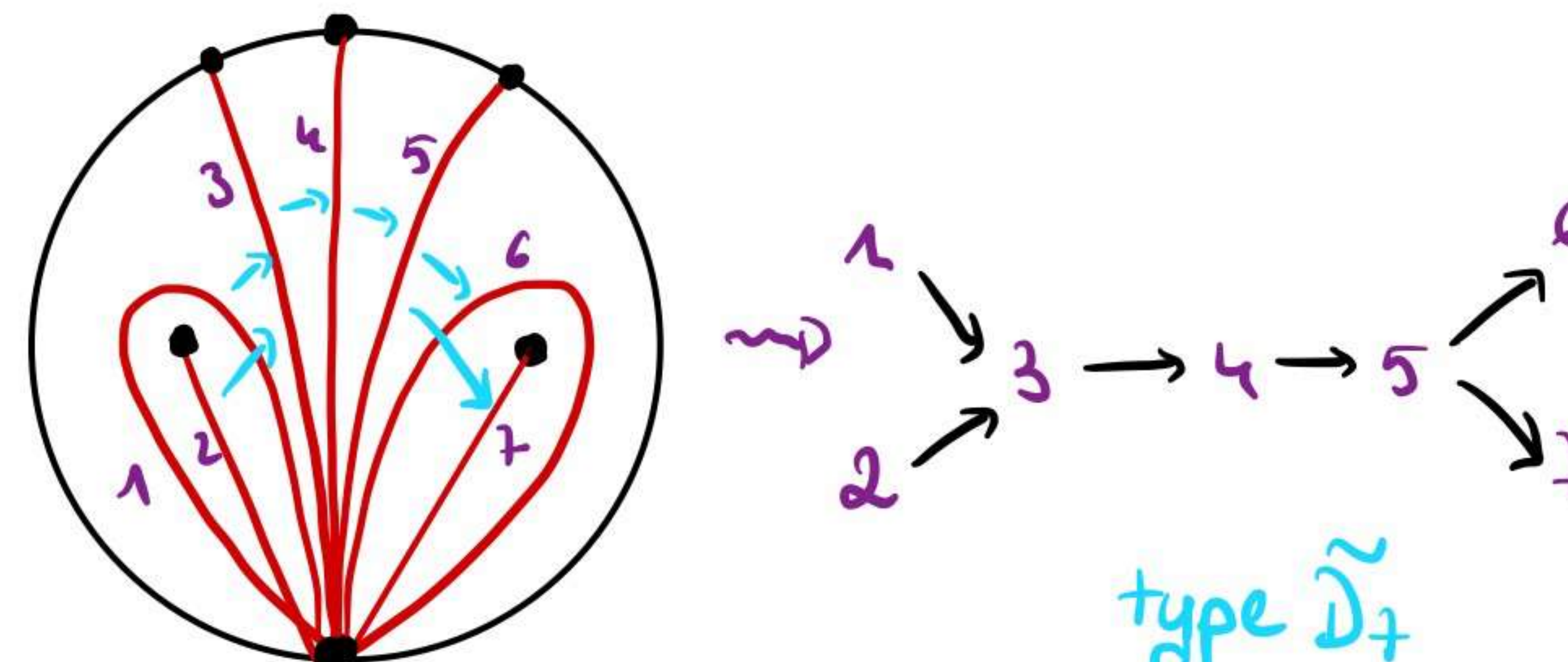

type $\tilde{D}_7$

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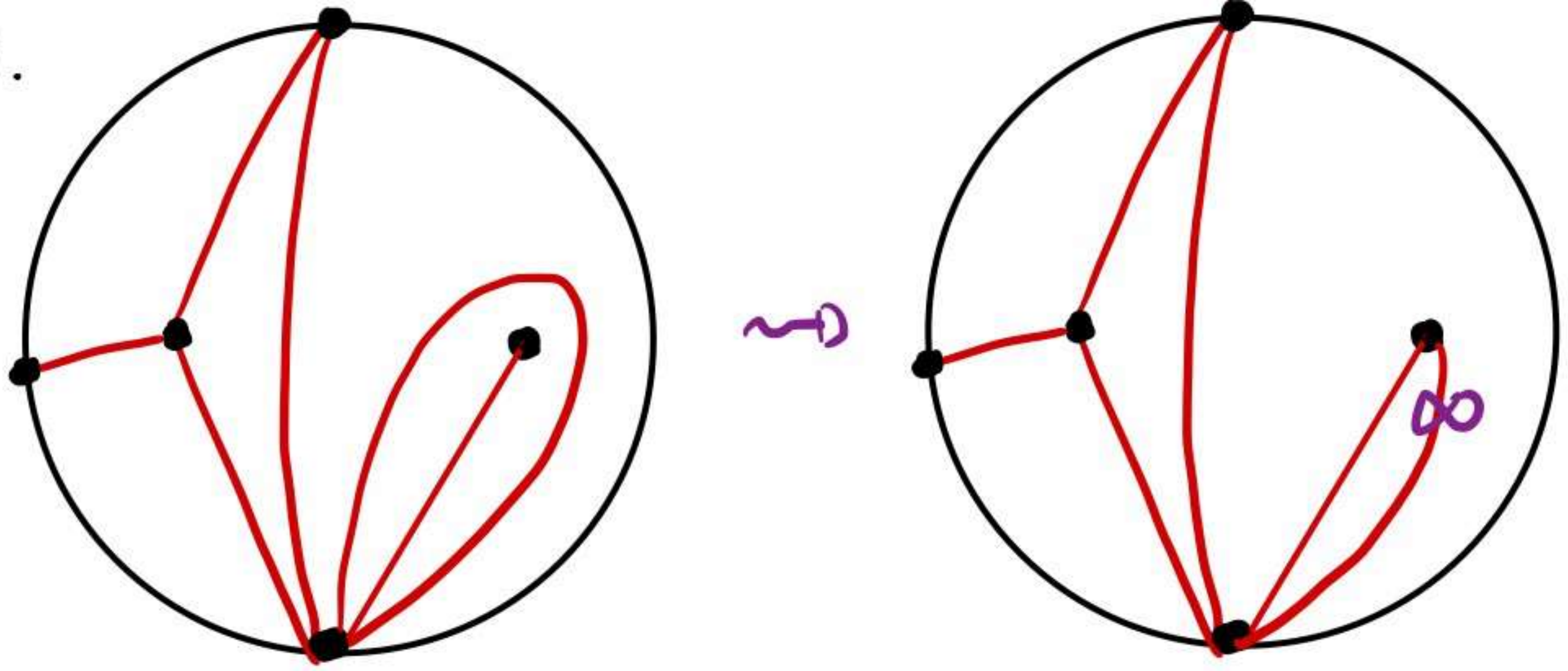

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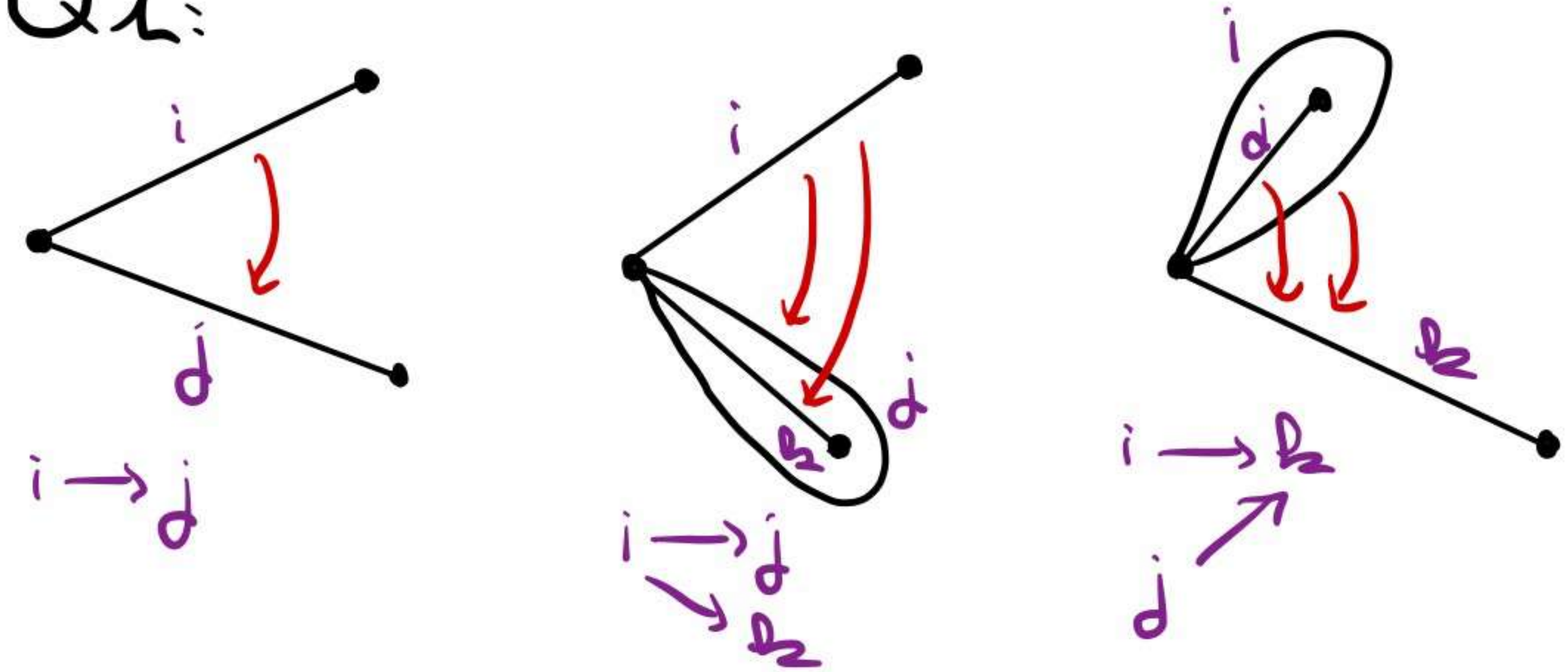

$i \to j$

$i \to j$, $i \to k$

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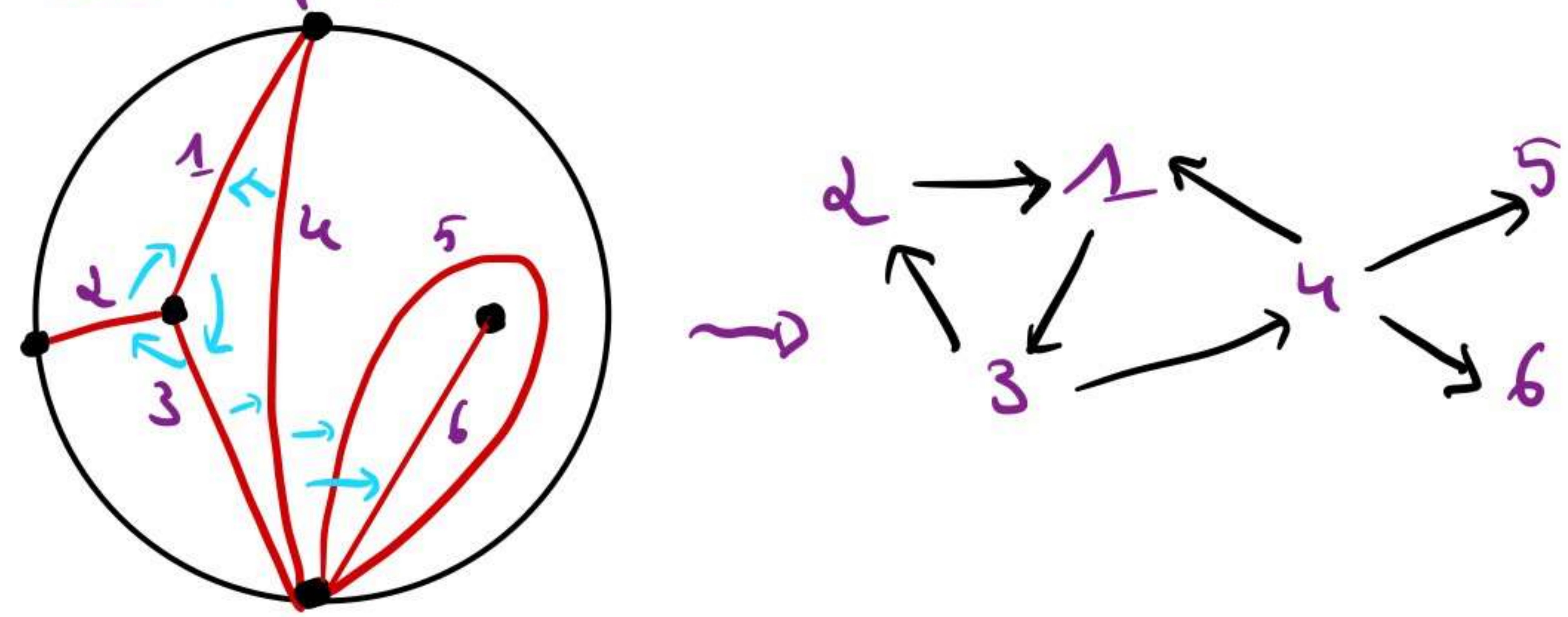
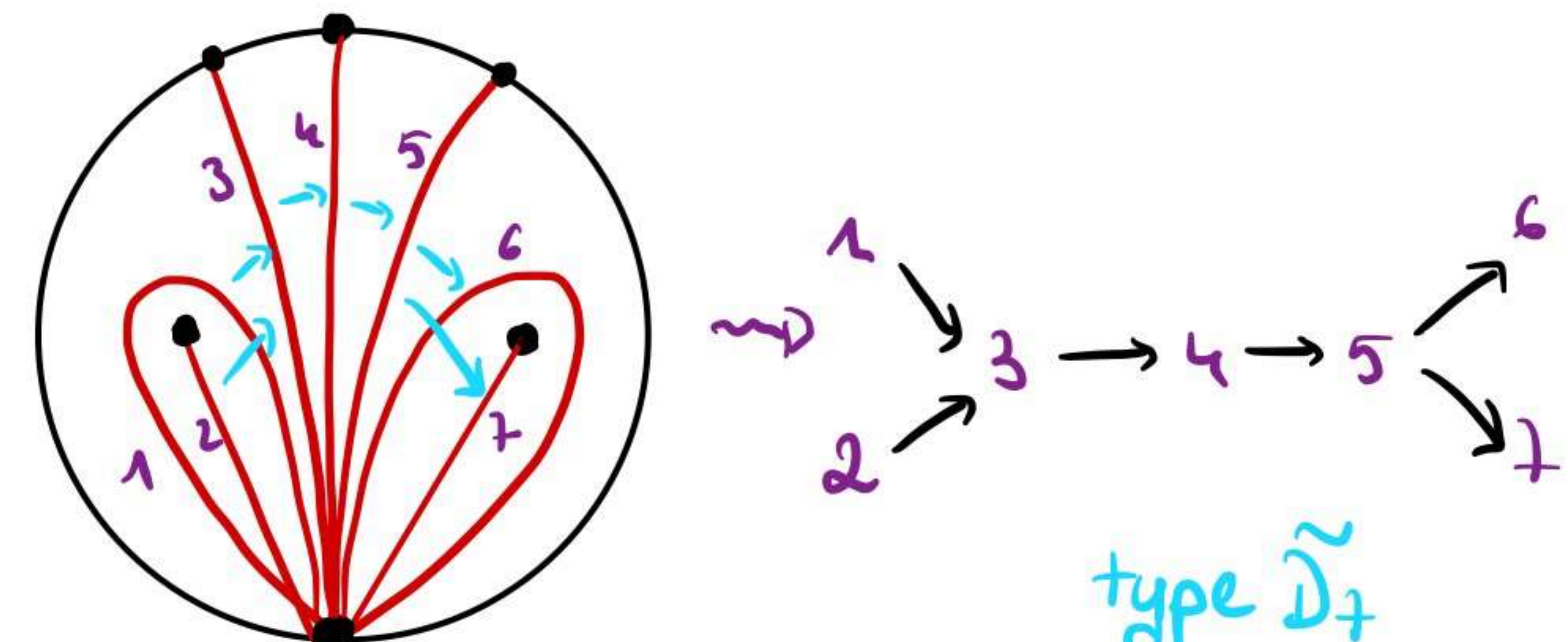


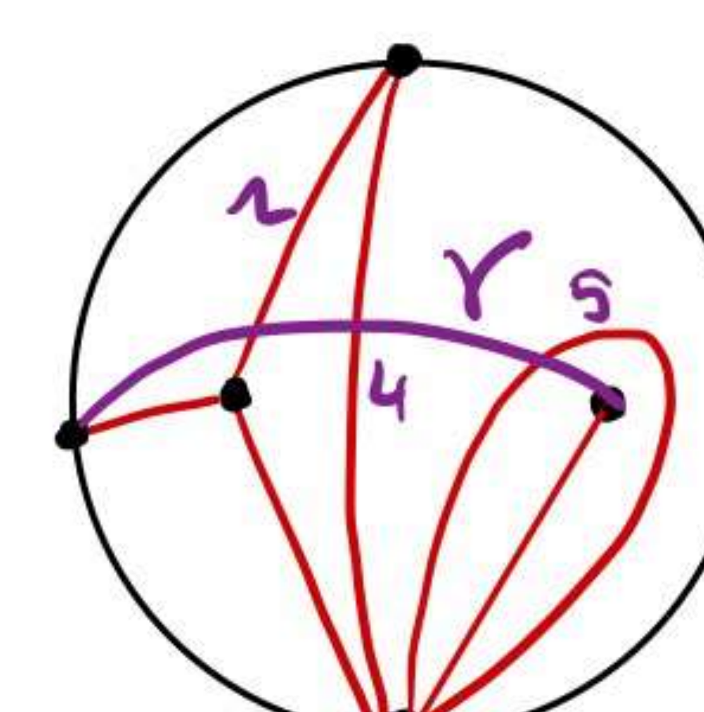


type $\tilde{D}_7$

- If $\gamma \notin T$, Q_γ is the full subquiver of Q_T of arcs which are crossed by γ.

Example:

$Q_\gamma = \; 1 \leftarrow 4 \rightarrow 5$



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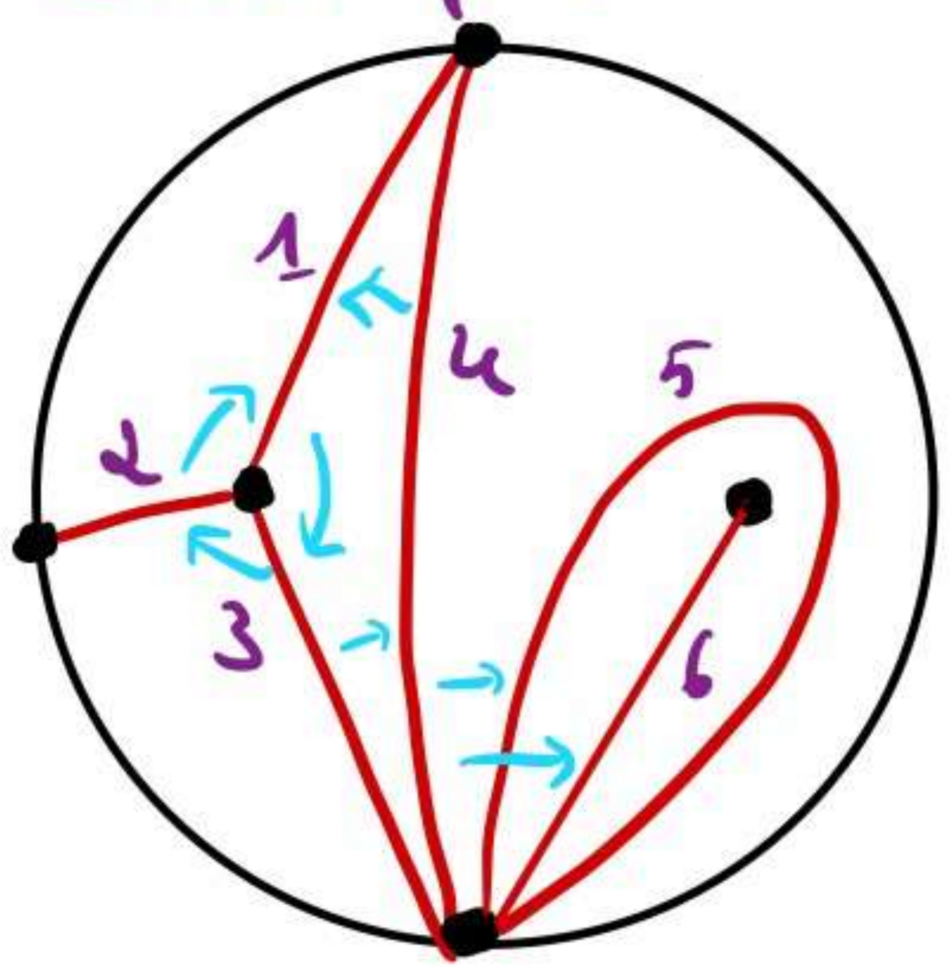
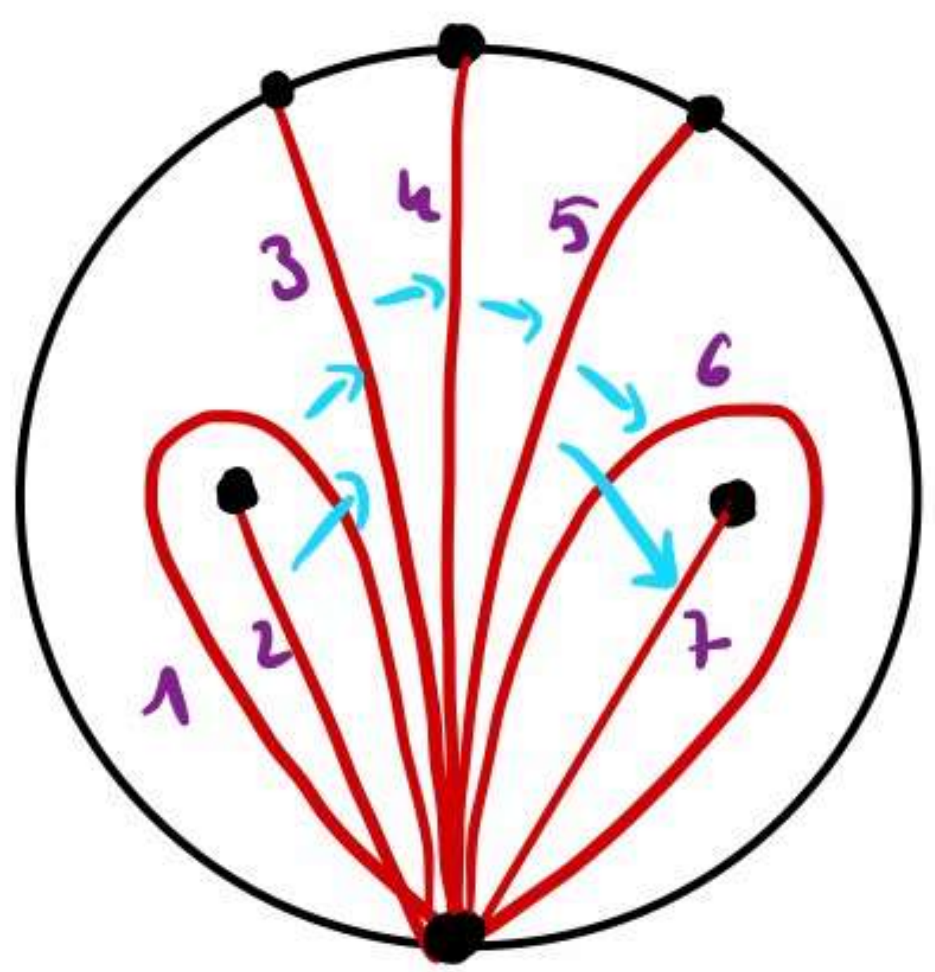


$$\begin{tikzcd} 2 \arrow[r] & 1 \arrow[dl] & & 5 \\ & & 4 \arrow[ul] \arrow[ur] \arrow[dr] & \\ & 3 \arrow[uul] \arrow[ur] & & 6 \end{tikzcd}$$



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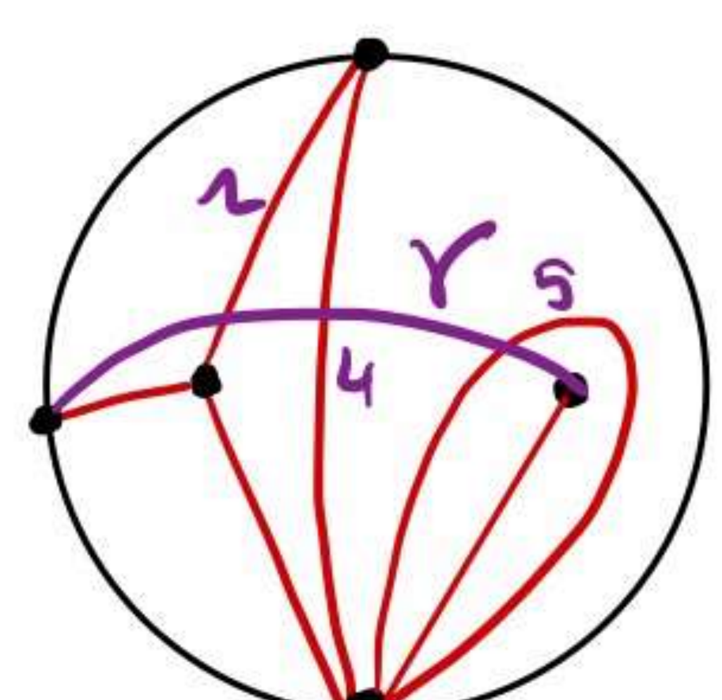
$$Q_\gamma = \quad 1 \leftarrow 4 \rightarrow 5$$



k : alg. closed field

- Quiver representations : $Q = (Q_0, Q_1)$

$$\operatorname{Rep} Q = \left\{ \left((V_i)_{i \in Q}, (f_\alpha)_{\alpha \in Q_1} \right) \right\}$$

e.g : $Q = 1 \longleftarrow 2$, $\operatorname{Rep} Q = \{ (V_1, V_2, f : V_2 \longrightarrow V_1) \}$.

Fix a dimension vector $\underline{v}$,

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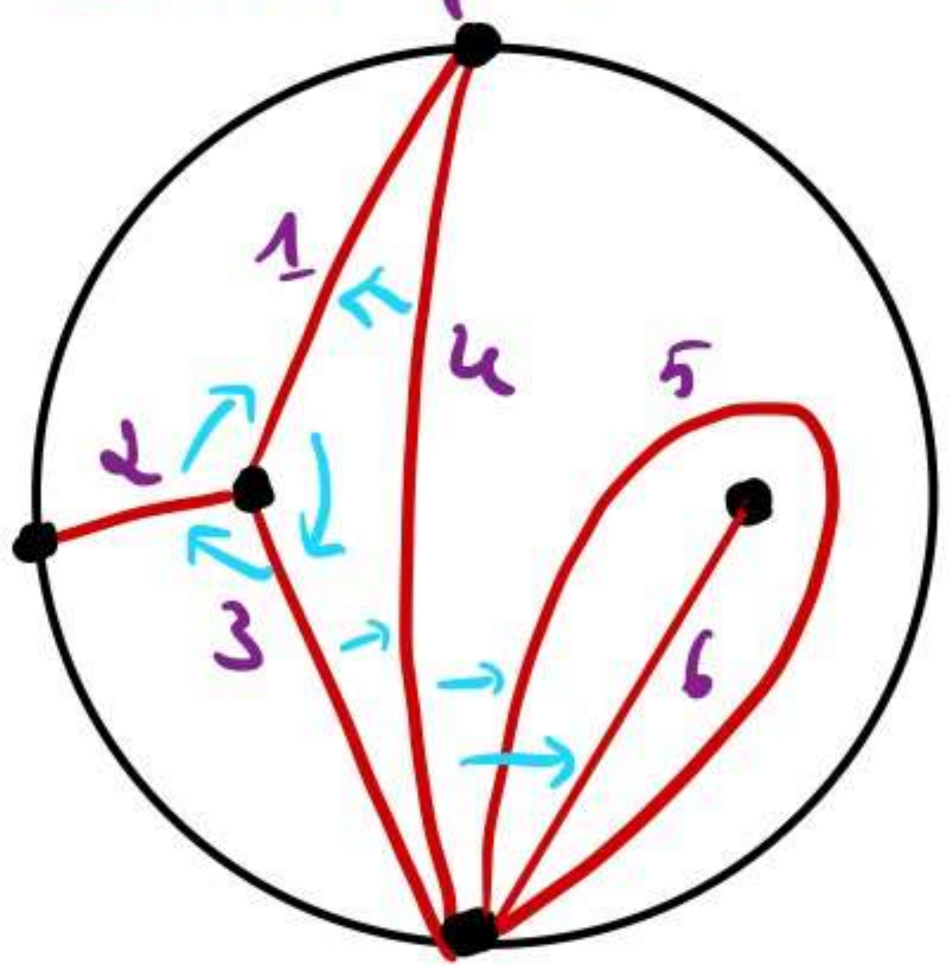
Then $\quad G_{\underline{v}} := \prod_{i \in Q_0} GL_{v_i}(k) \curvearrowright \operatorname{Rep}(Q, \underline{v})$.

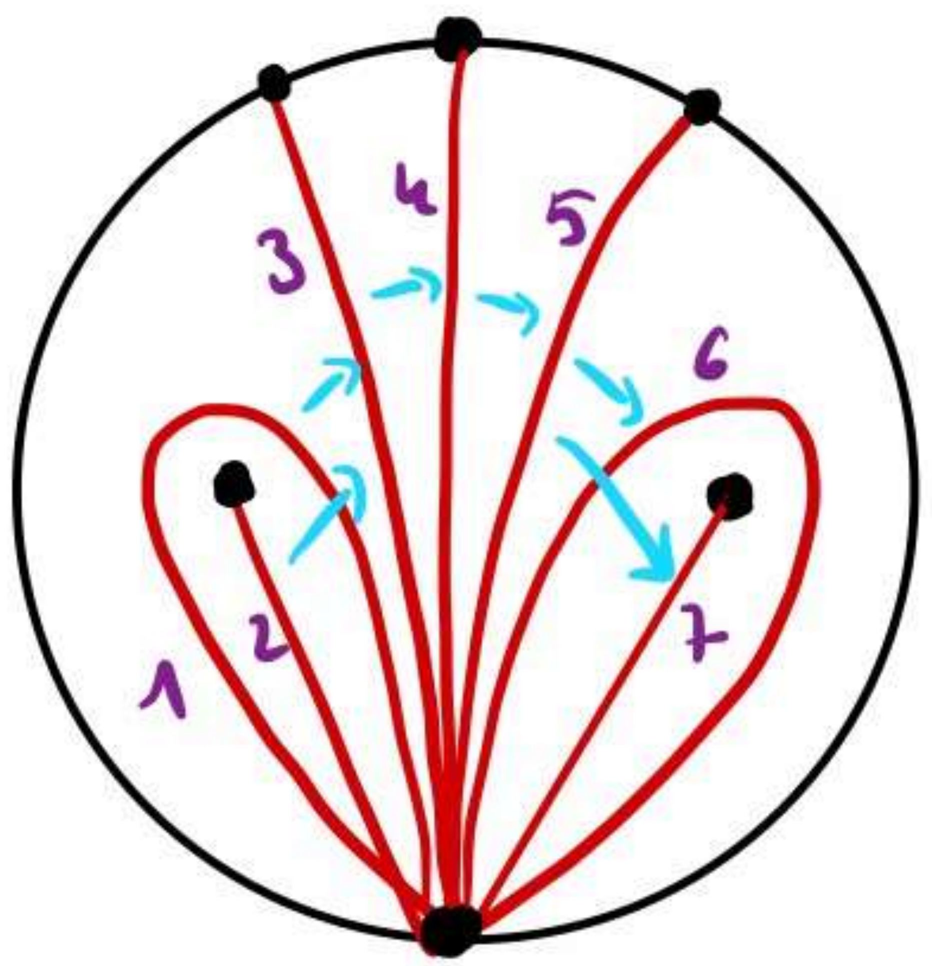
$\Rightarrow$ Desciption of $G_{\underline{v}}$-orbits : types A, D, E or $\tilde{A}, \tilde{D}, \tilde{E}$.

quiver varieties

Examples:



$\rightsquigarrow$
$$\begin{tikzcd} 2 \arrow[r] & 1 \arrow[dl] & & 5 \\ 3 \arrow[u] \arrow[rr] & & 4 \arrow[ul] \arrow[ur] \arrow[dr] & \\ & & & 6 \end{tikzcd}$$



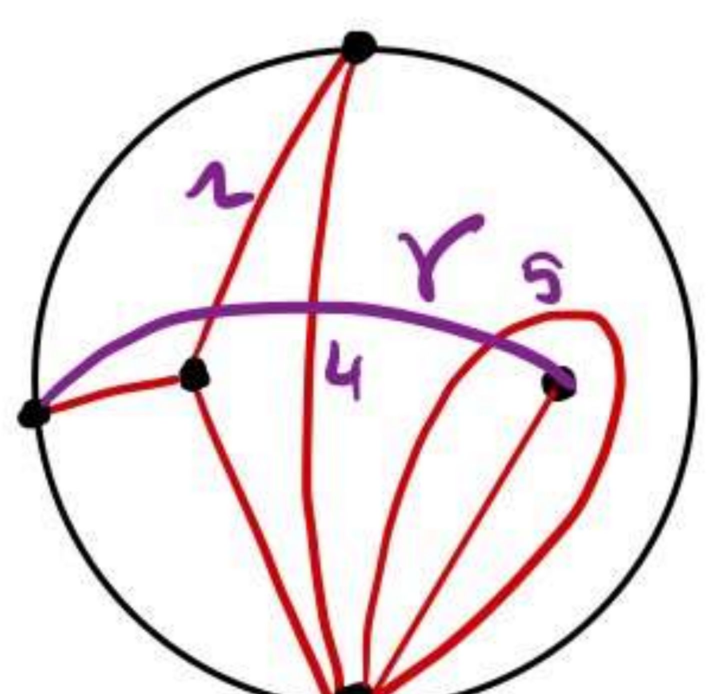
$\rightsquigarrow$
$$\begin{tikzcd} 1 \arrow[dr] & & & & 6 \\ & 3 \arrow[r] & 4 \arrow[r] & 5 \arrow[ur] \arrow[dr] & \\ 2 \arrow[ur] & & & & 7 \end{tikzcd}$$

type $\tilde{D}_7$

- If $\gamma \notin T$, Q_γ is the full subquiver of Q_T of arcs which are crossed by γ.

Example:

$$Q_\gamma = 1 \leftarrow 4 \rightarrow 5$$



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quiver varieties

V' is a Q-subrepresentation of V:
- $\forall i \in Q_0$, $V_i' \subset V_i$
- $\forall \alpha : i \to j \in Q_1$, $f_\alpha(V_i') \subset V_j'$.

Example: $\mathbb{C} \xleftarrow{id} \mathbb{C}$ has 3 subrepresentations:

$$\{0\} \longleftarrow \{0\} \; ; \; \mathbb{C} \longleftarrow \{0\} \; ; \; \mathbb{C} \xleftarrow{id} \mathbb{C}$$

$\Bbbk$: alg. closed field

- **Quiver representations:** $Q = (Q_0, Q_1)$

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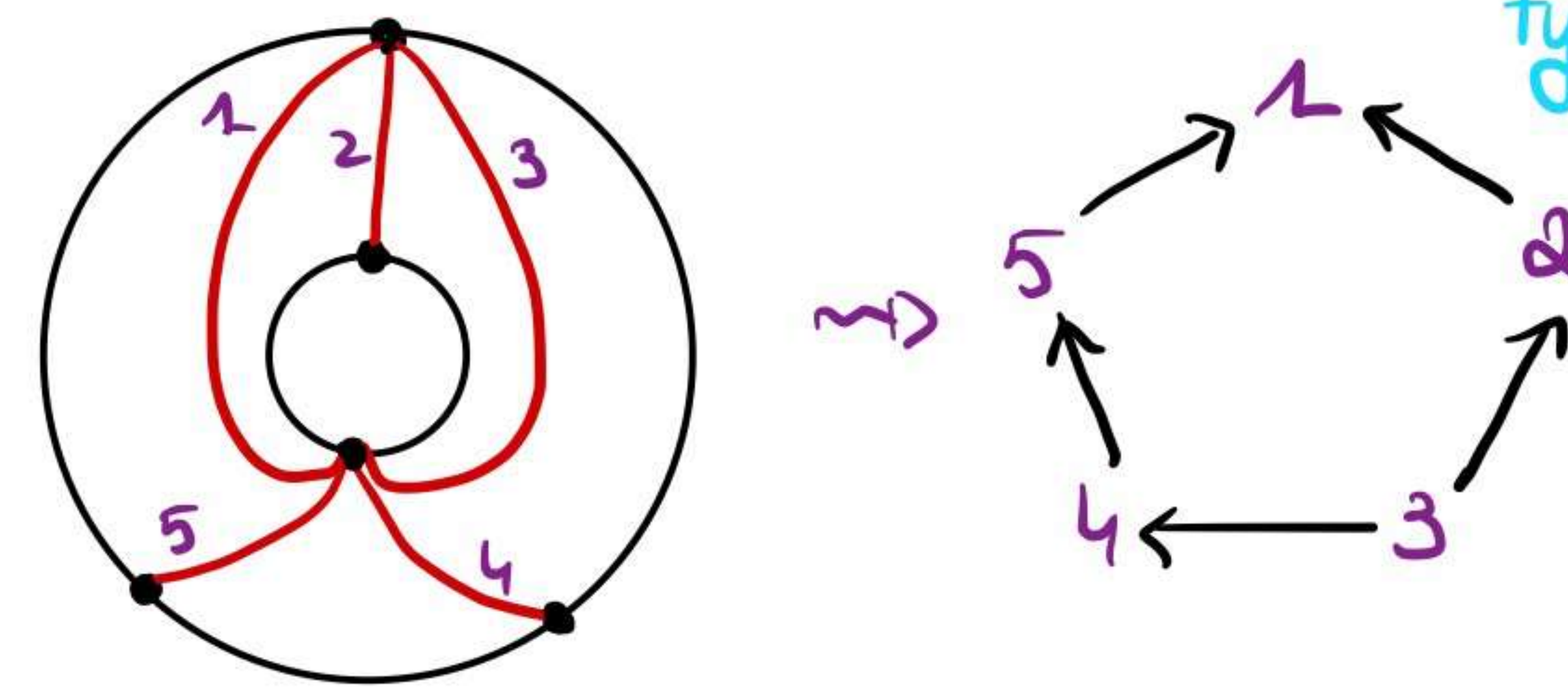
- **Frieze from a triangulated annulus:**

type $\tilde{A}_5$



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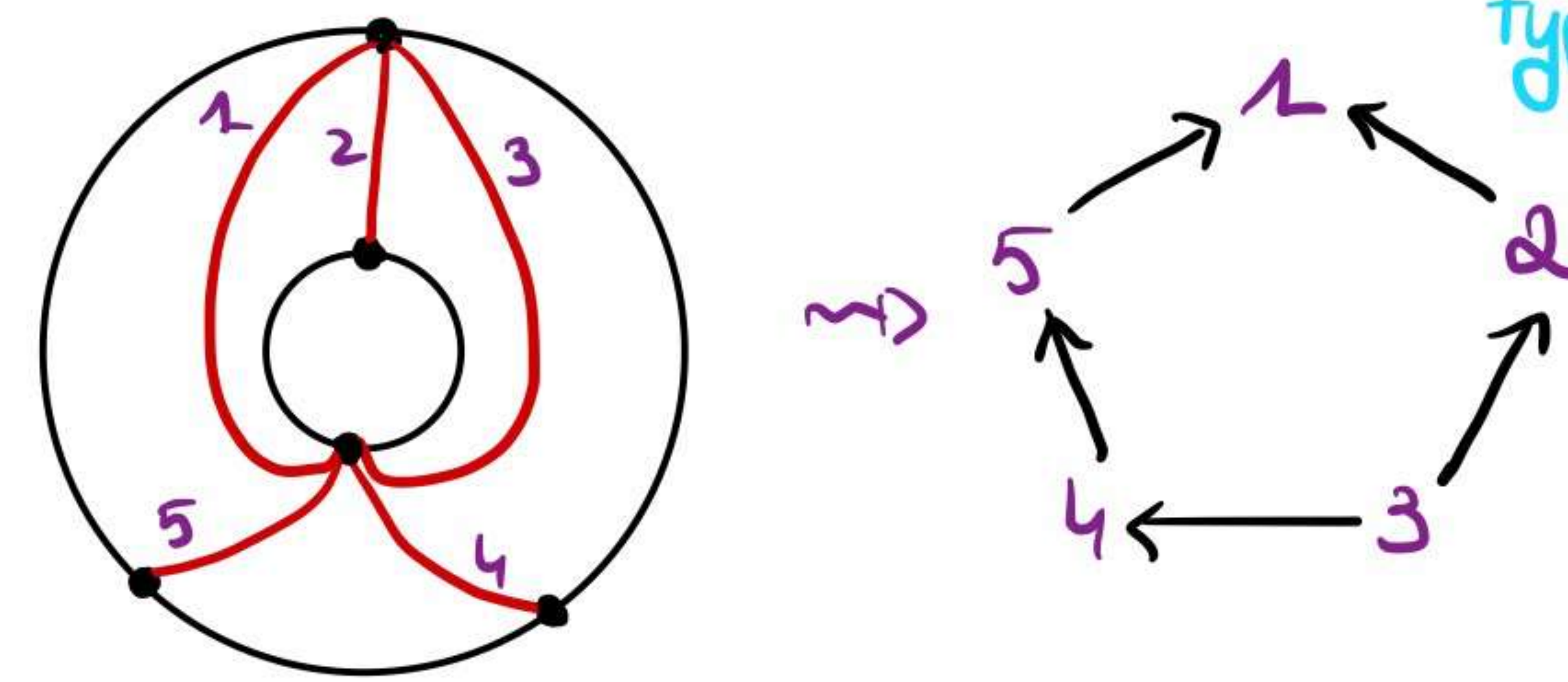
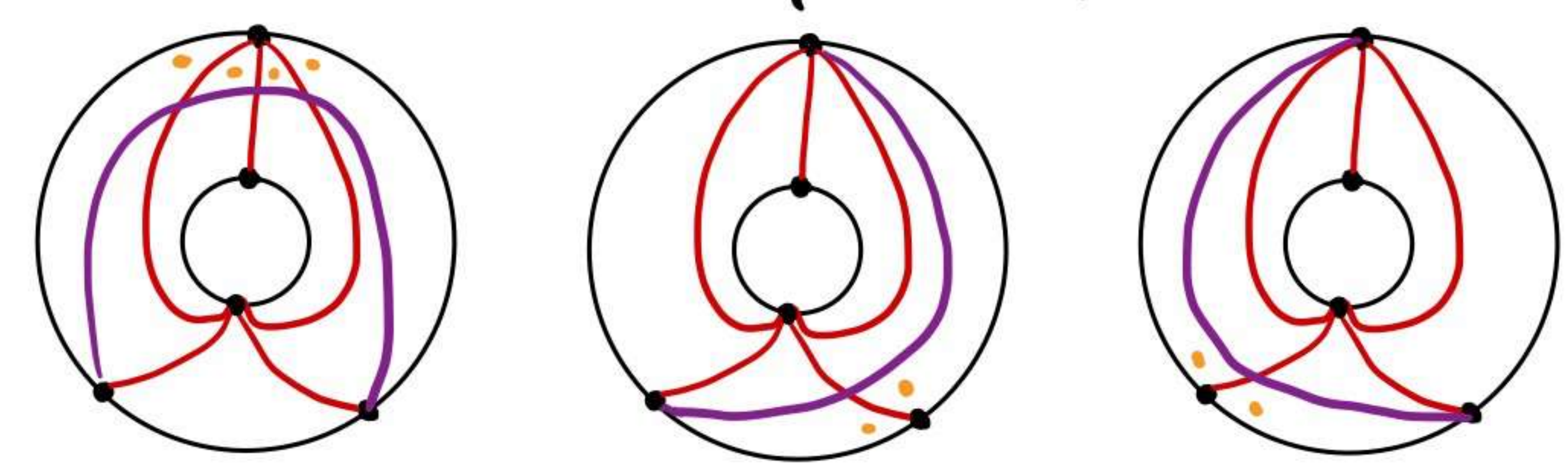


"Counting adjacent triangles" = number of submodules of a quiver of an arc

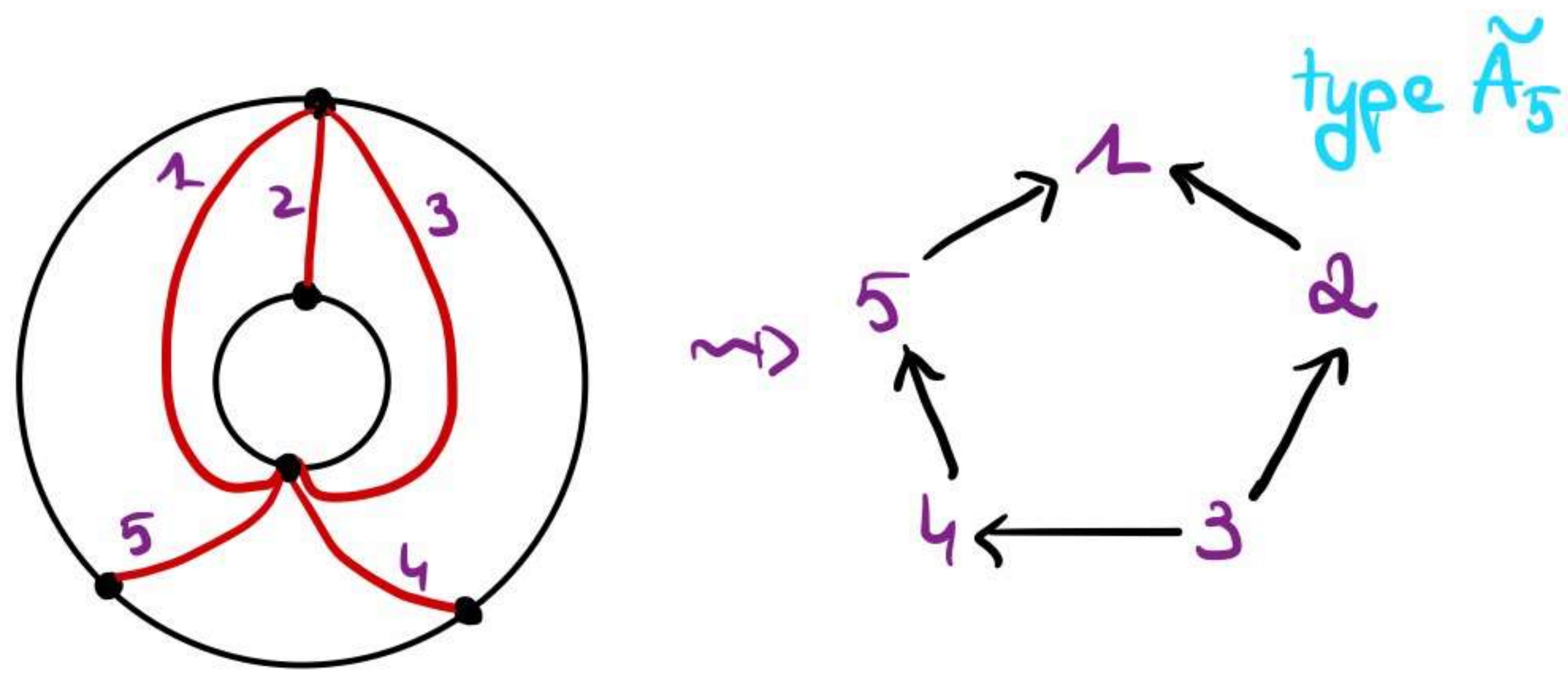


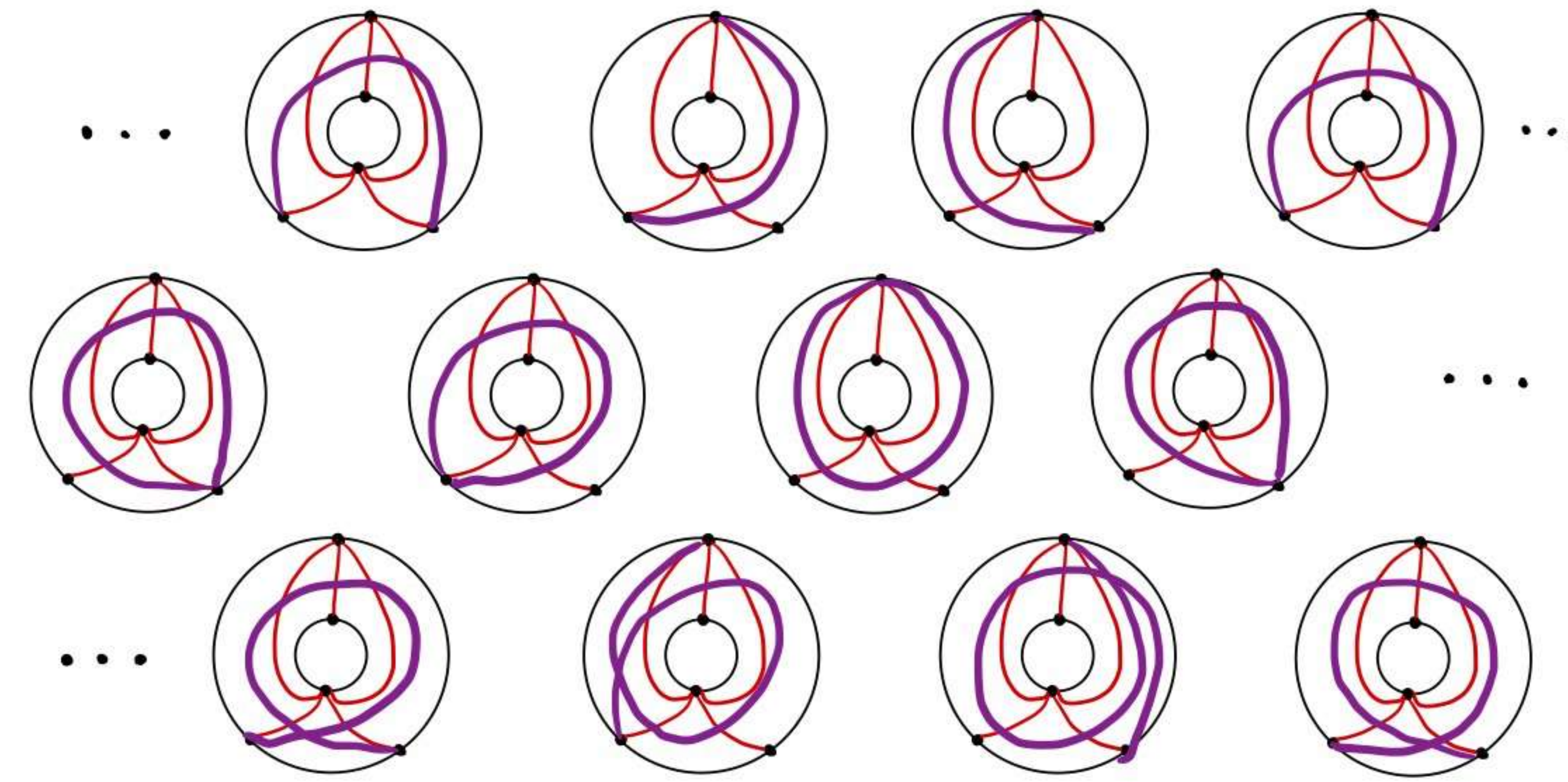
$1 \leftarrow 2 \leftarrow 3$: $\boxed{4}$ submodules

4 : $\boxed{2}$ submodules

5 : $\boxed{2}$ submodules

- Frieze from a triangulated annulus:





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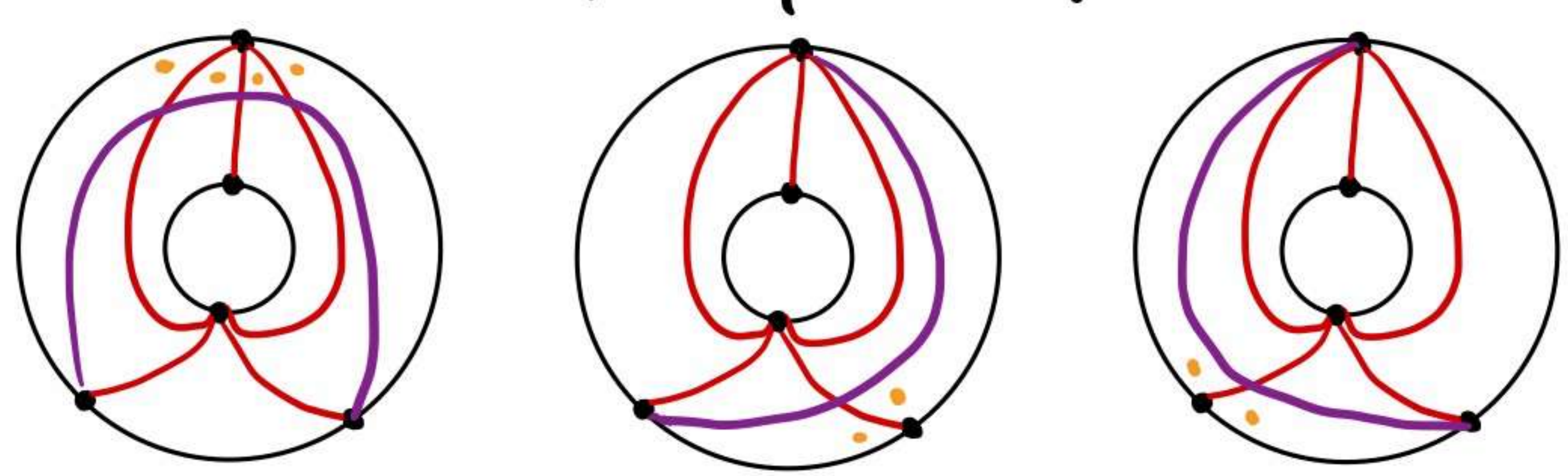


$1 \leftarrow 2 \leftarrow 3$

$\boxed{4}$ submodules

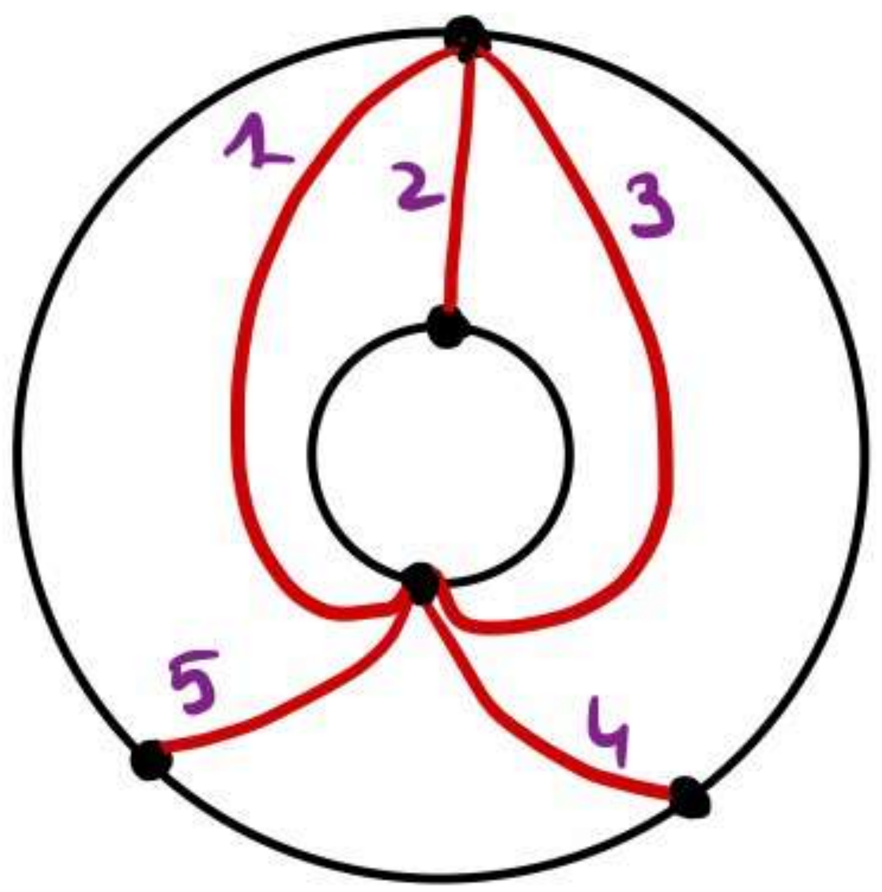
$\overset{4}{\bullet}$

$\boxed{2}$ submodules

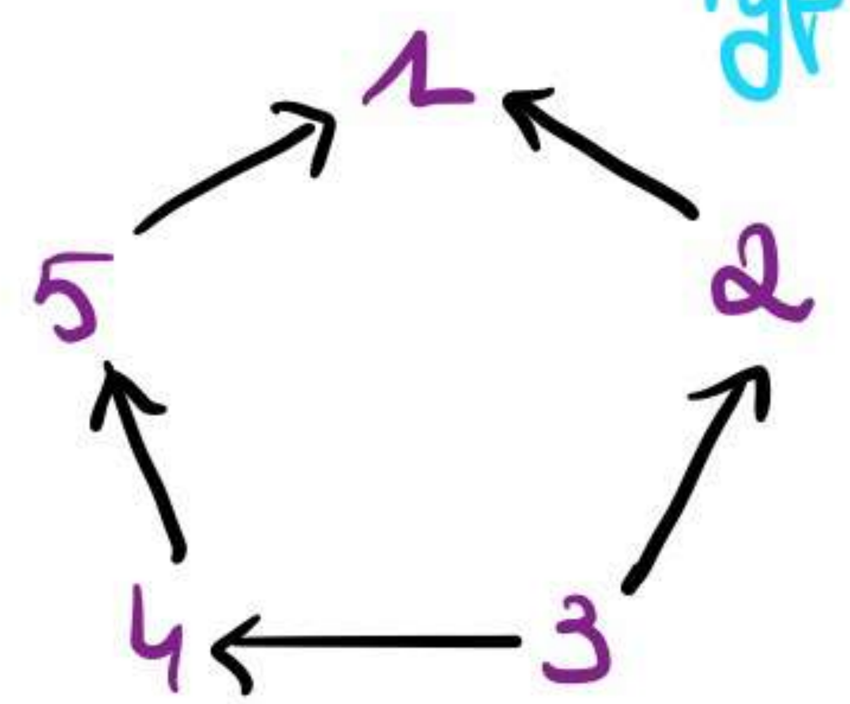
$\overset{5}{\bullet}$

$\boxed{2}$ submodules

- **Frieze from a triangulated annulus:**



$\rightsquigarrow$



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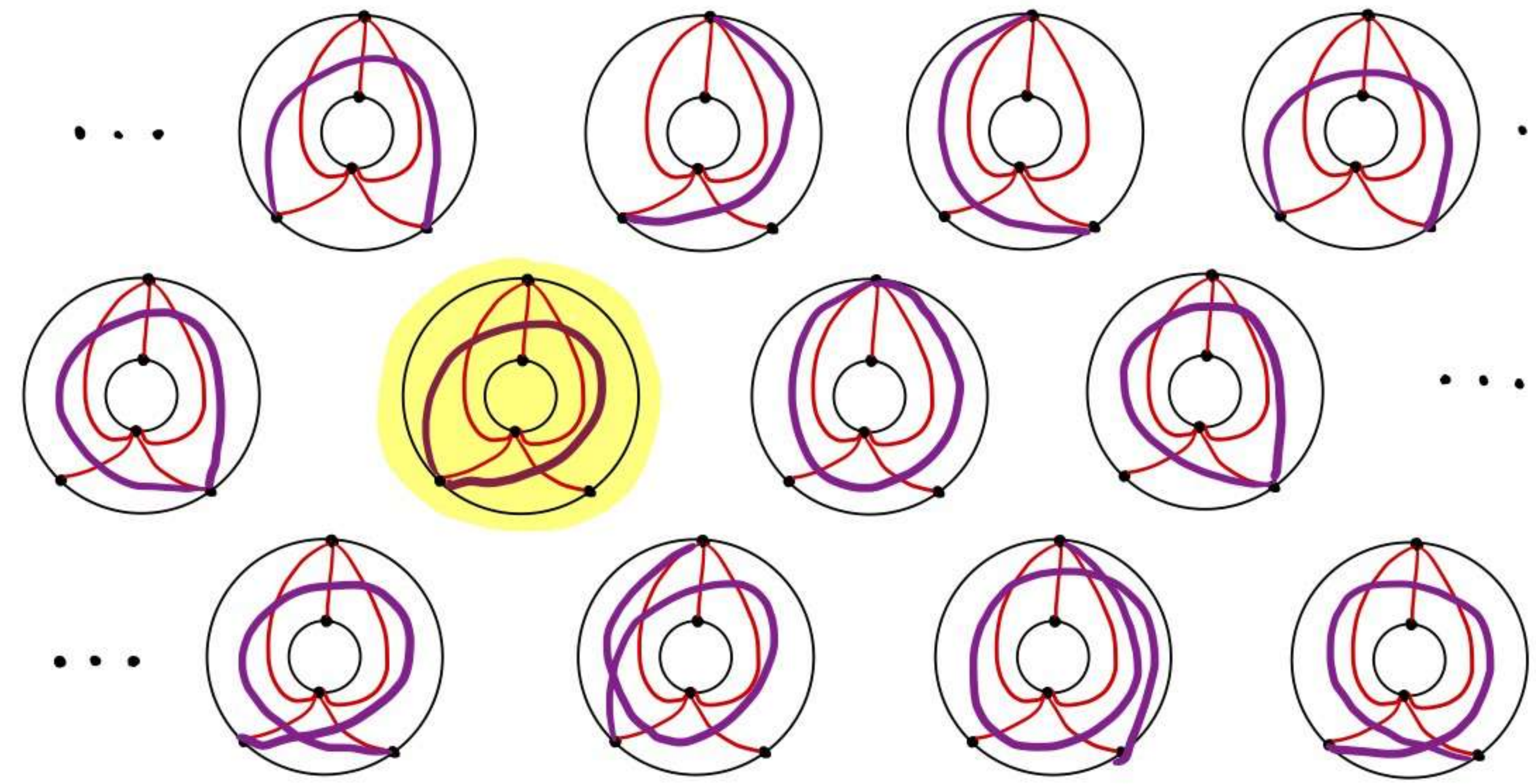
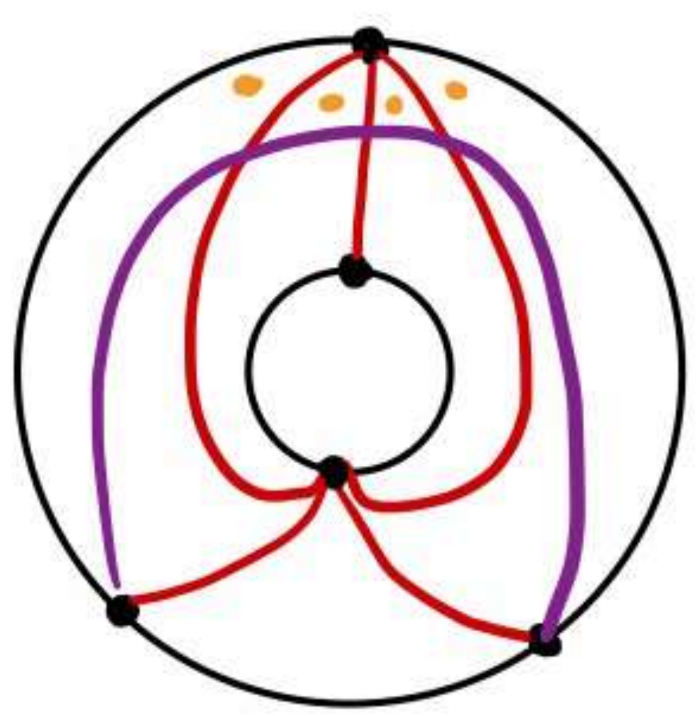


1		1	1	1 ...
... 4	2		2	4
7	7	3	7 ...	
... 12	10		10	12

$Q_\gamma = 1 \leftarrow 2 \leftarrow 3 \rightarrow 4$

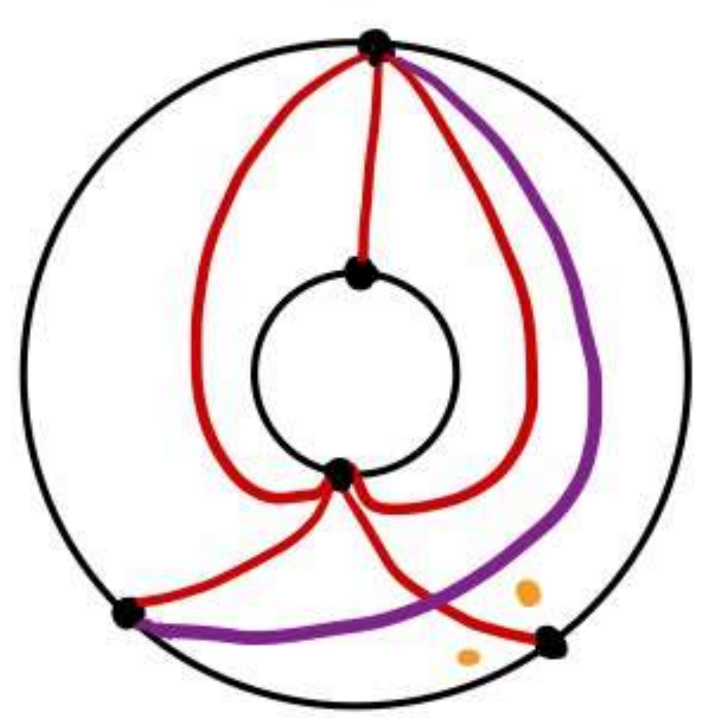
$\emptyset,\ 1,\ 4,\ 1\ 4,\ 1 \leftarrow 2,\ 1 \leftarrow 2\ \ 4,\ Q_\gamma$

"Counting adjacent triangles" = number of submodules of a quiver of an arc



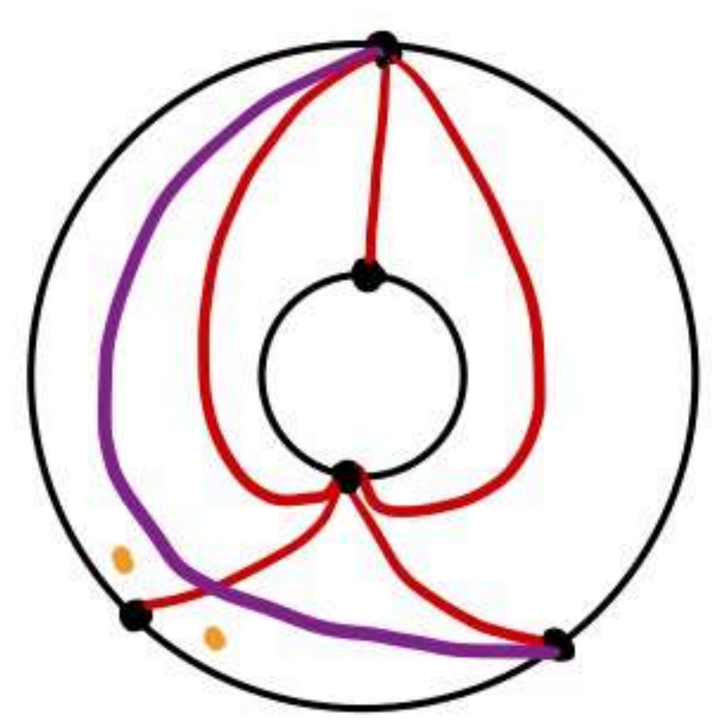
$1 \leftarrow 2 \leftarrow 3$

$\boxed{4}$ submodules



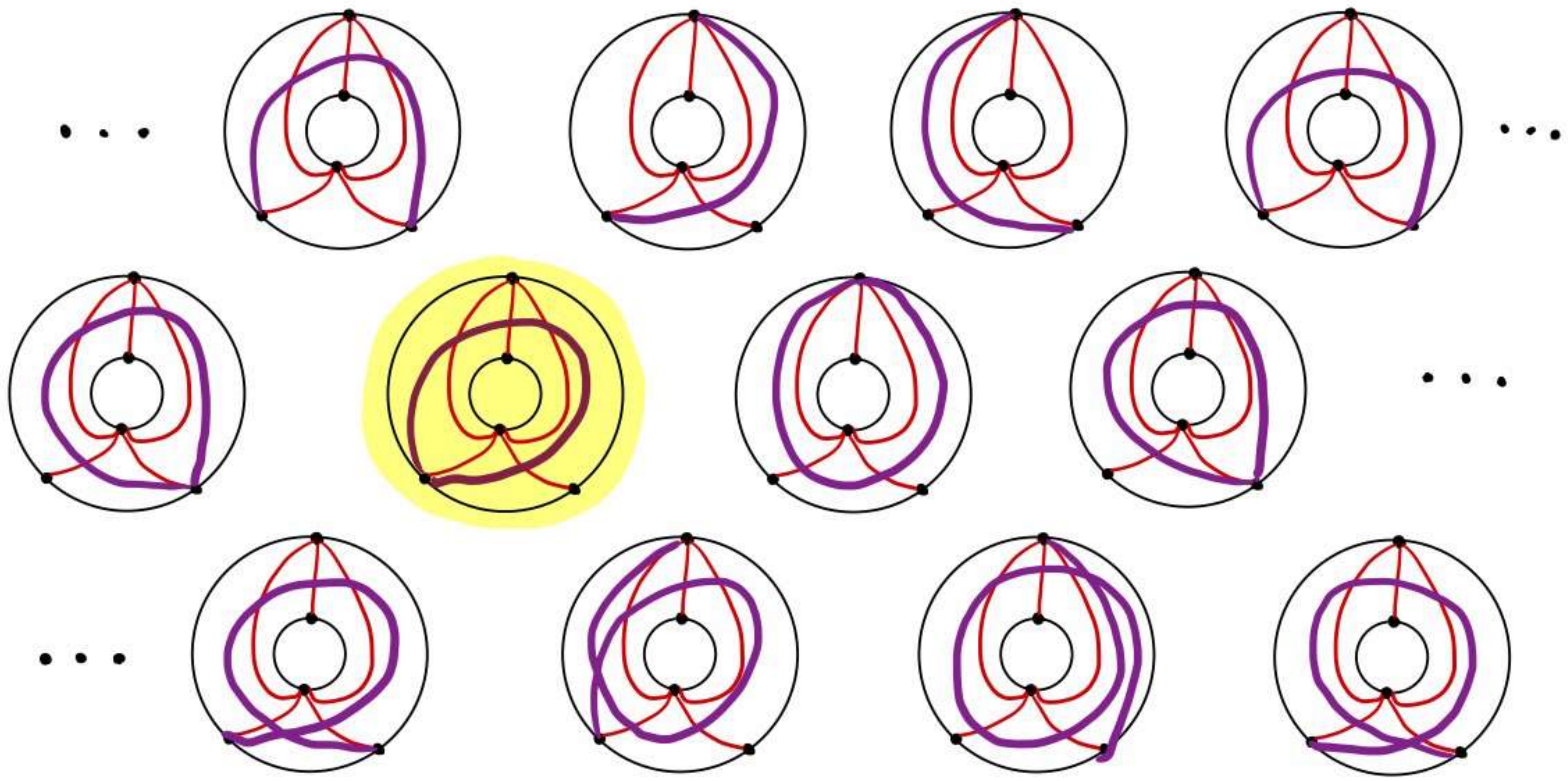
4

$\boxed{2}$ submodules



5

$\boxed{2}$ submodules



1		1		1		1		…	
…	4		2		2		4		
7		7		3		7		…	
…	12		10		10		12		

$Q_\gamma = 1 \leftarrow 2 \leftarrow 3 \rightarrow 4$

$\emptyset$, 1, 4, $1\ \ 4$, $1 \leftarrow 2$, $1 \leftarrow 2\ \ 4$, Q_γ

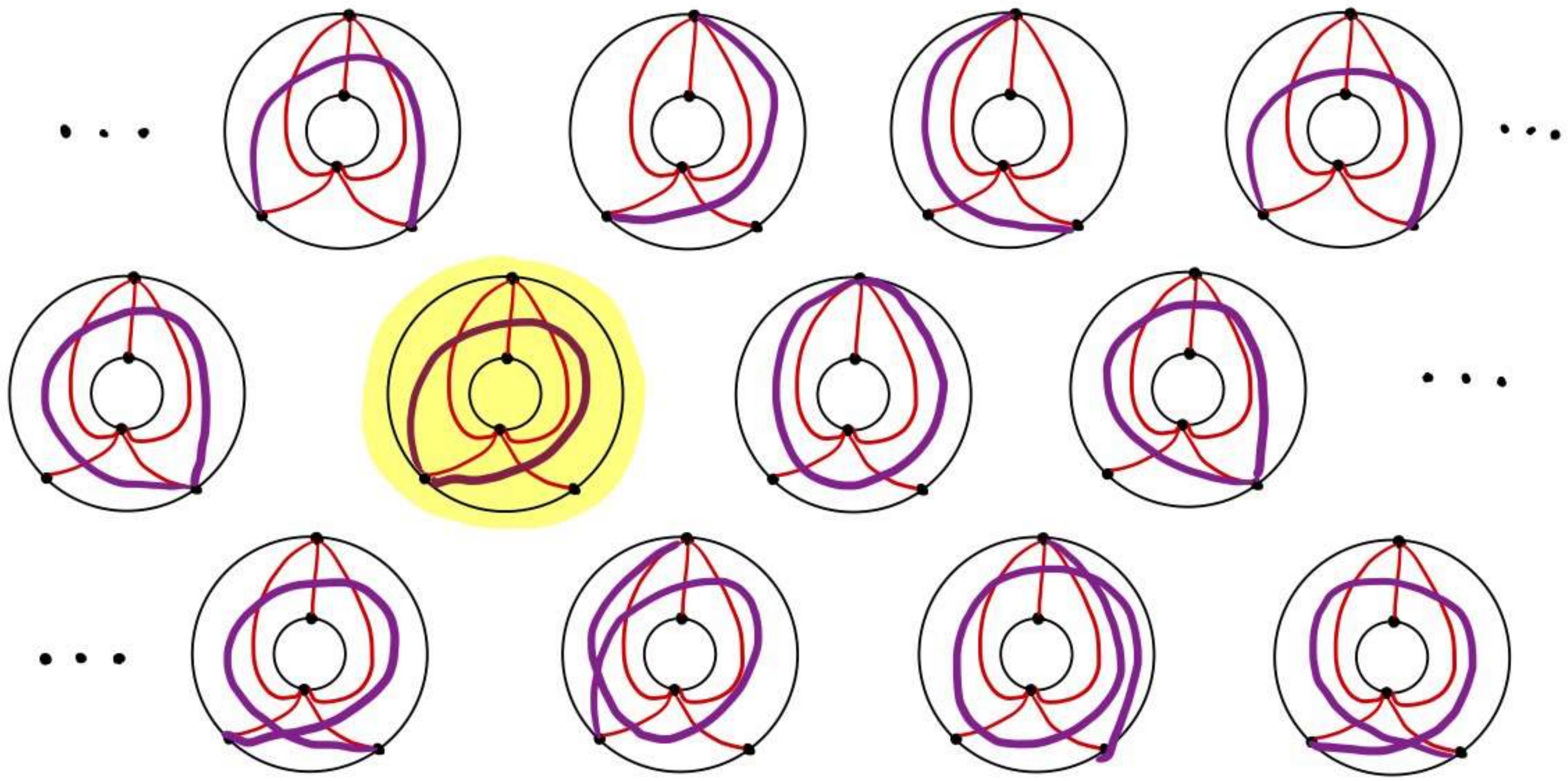
2) Cluster category of affine type D:

Q: quiver, orientation of $\tilde{D}_{n+1}$ (or mutation equivalent)

We have: $\operatorname{Rep} Q \simeq \operatorname{mod} \Bbbk Q$

finite dim. $\Bbbk$-representations of Q

finitely generated modules over $\Bbbk Q$



$\Downarrow$

1	1	1	1	...
... 4	2	2	4	
7	7	3	7	...
... 12	10	10	12	

$Q_\gamma = 1 \leftarrow 2 \leftarrow 3 \rightarrow 4$

$\emptyset,\ \overset{\bullet}{1},\ \overset{\bullet}{4},\ \overset{\bullet}{1}\ \overset{\bullet}{4},\ 1 \leftarrow 2,\ 1 \leftarrow 2\ \overset{4}{\bullet},\ Q_\gamma$

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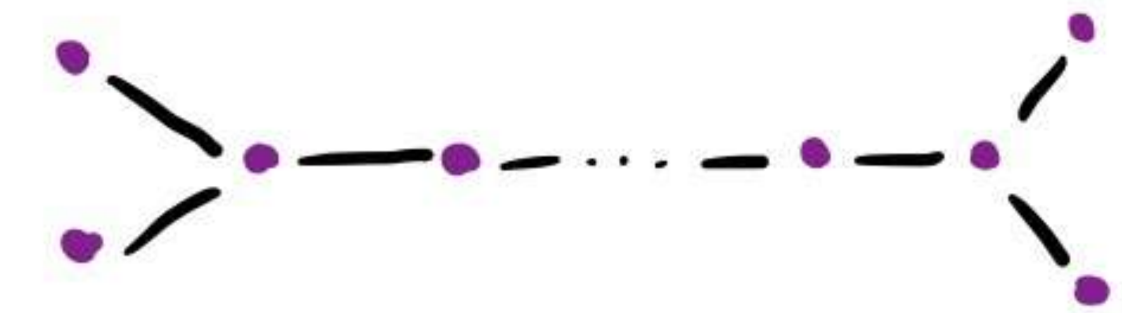


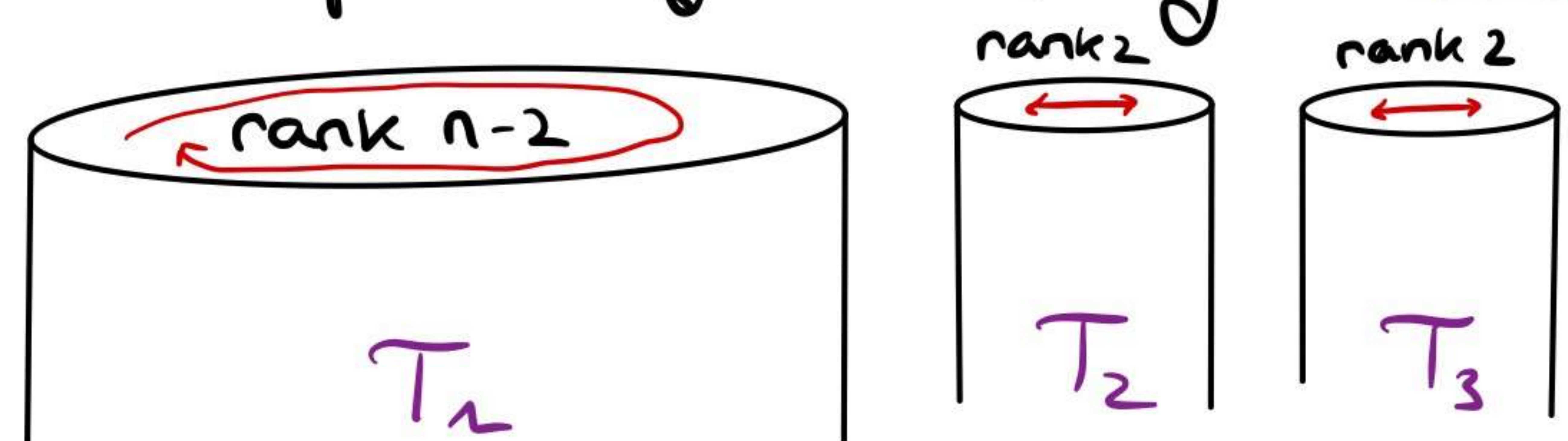
We have:

$$\underbrace{\operatorname{Rep} Q}_{\substack{\text{finite dim.} \\ k\text{-representations of } Q}} \simeq \underbrace{\operatorname{mod} kQ}_{\substack{\text{finitely generated} \\ \text{modules over } kQ}}$$

Auslander-Reiten quiver: (AR-quiver)

$\Big($ indecomposable objects (vertices) , irreducible morphisms (arrows) $\Big)$

Indecomposable objects are arranged in tubes.



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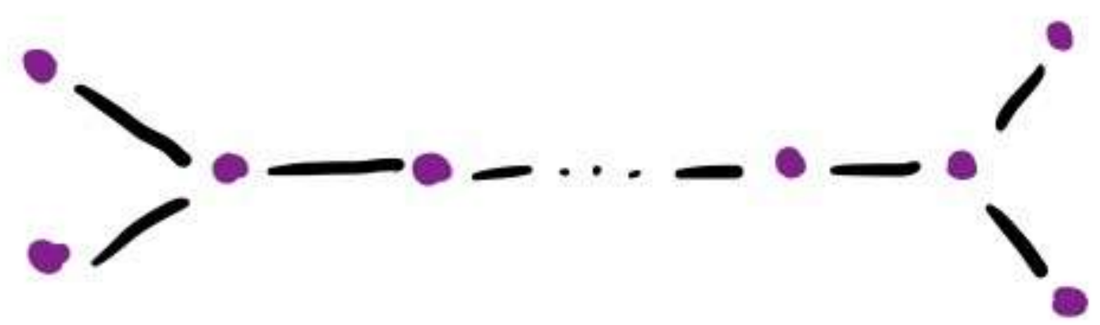

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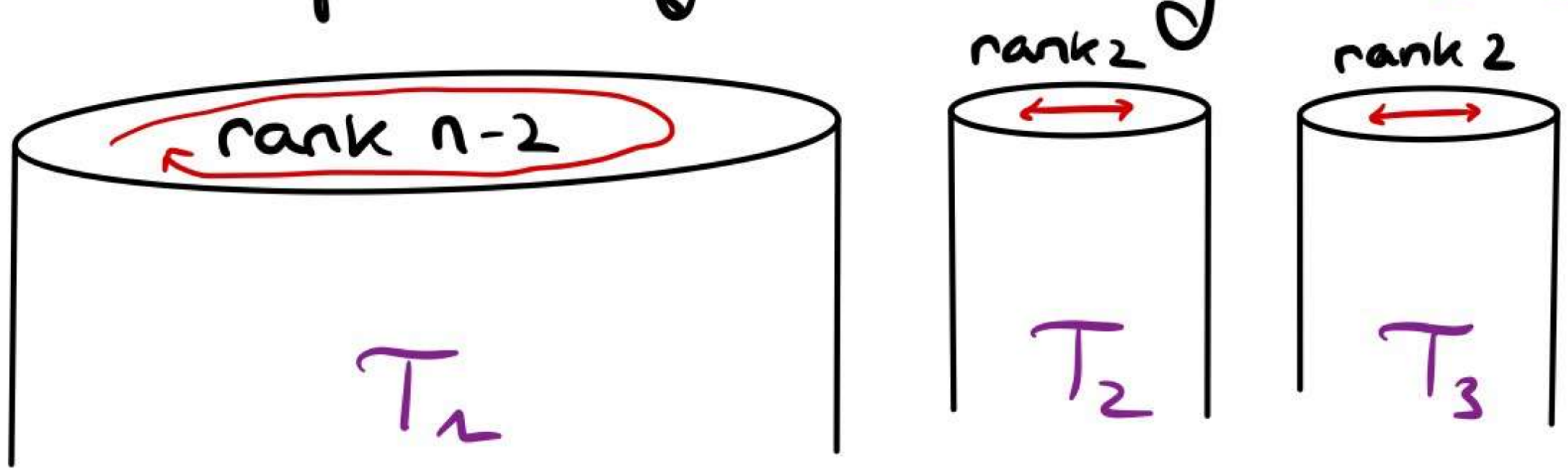

- Same for cluster category:

$$\mathcal{C}_Q := D^b(\mathrm{mod}\,kQ) / \tau^{-1}[1]$$

[Buan-Marsh-Reineke-Reiten-Todorov, 06]

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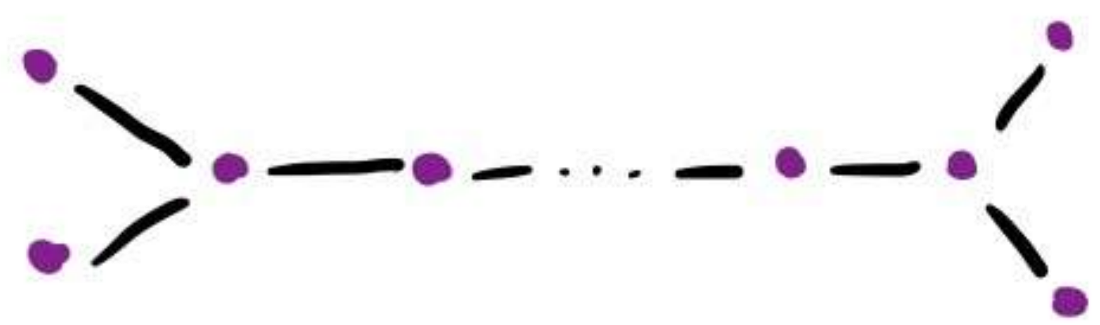

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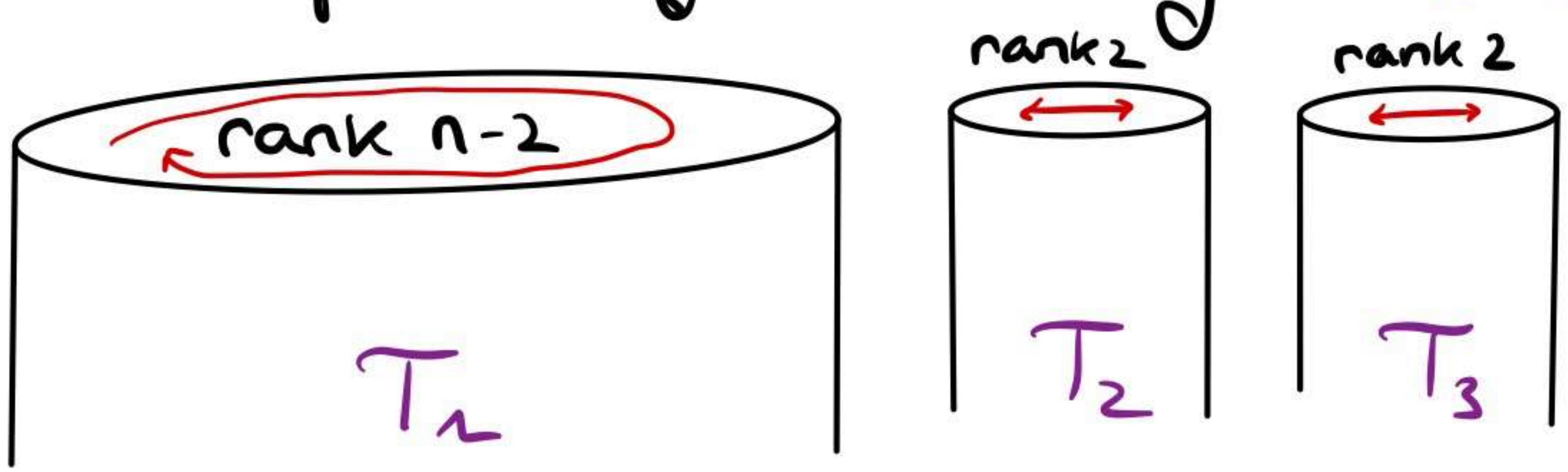

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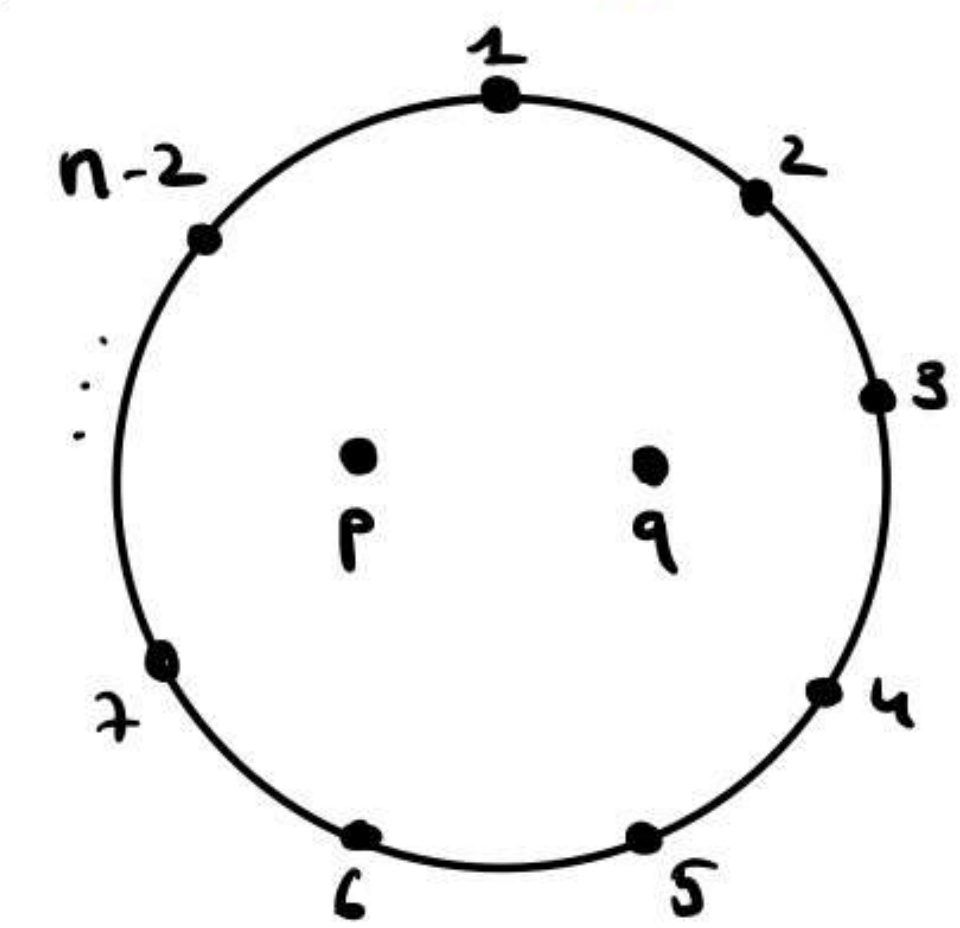
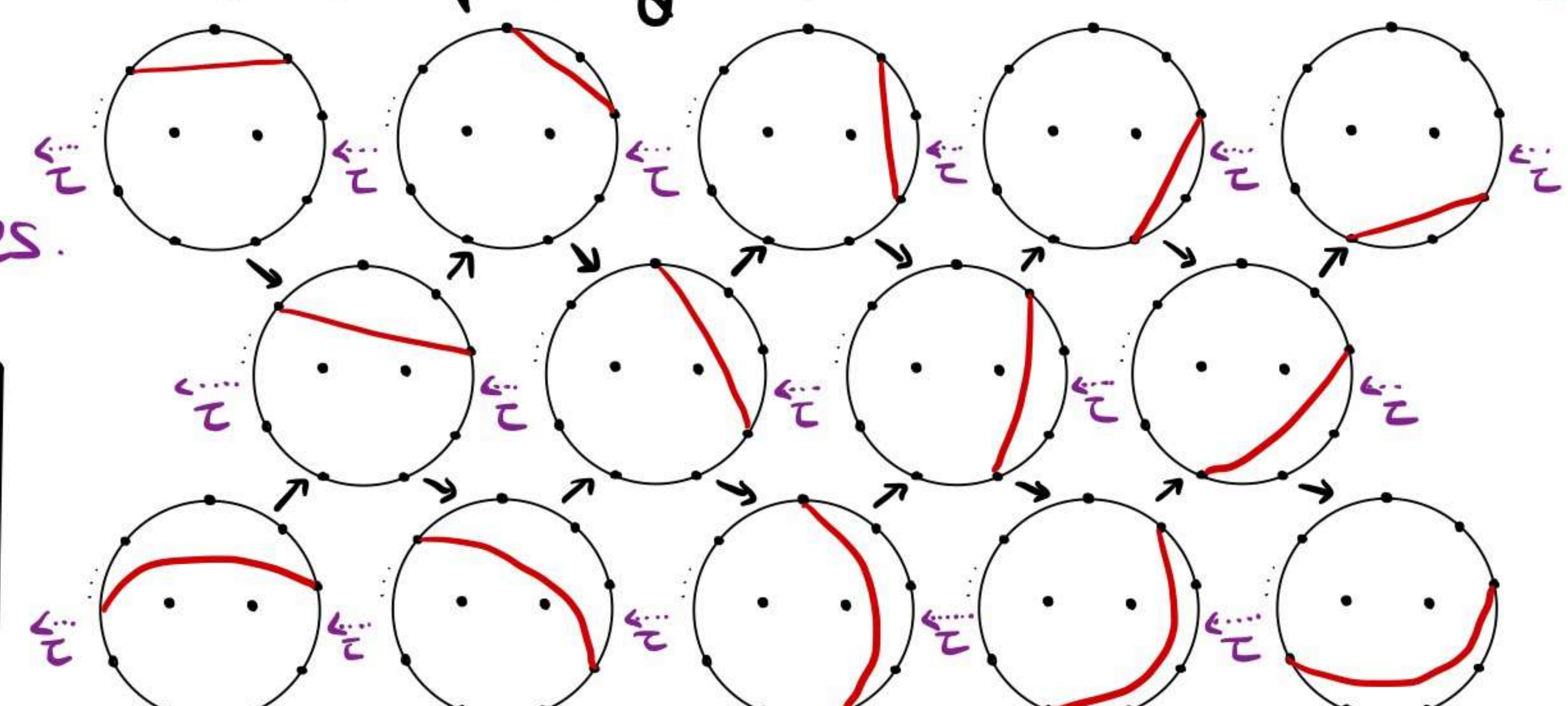
[Fomin-Shapiro-Thurston, 08] This cluster category has a surface model:



- Arcs corresponding to modules in the tubes: T_1



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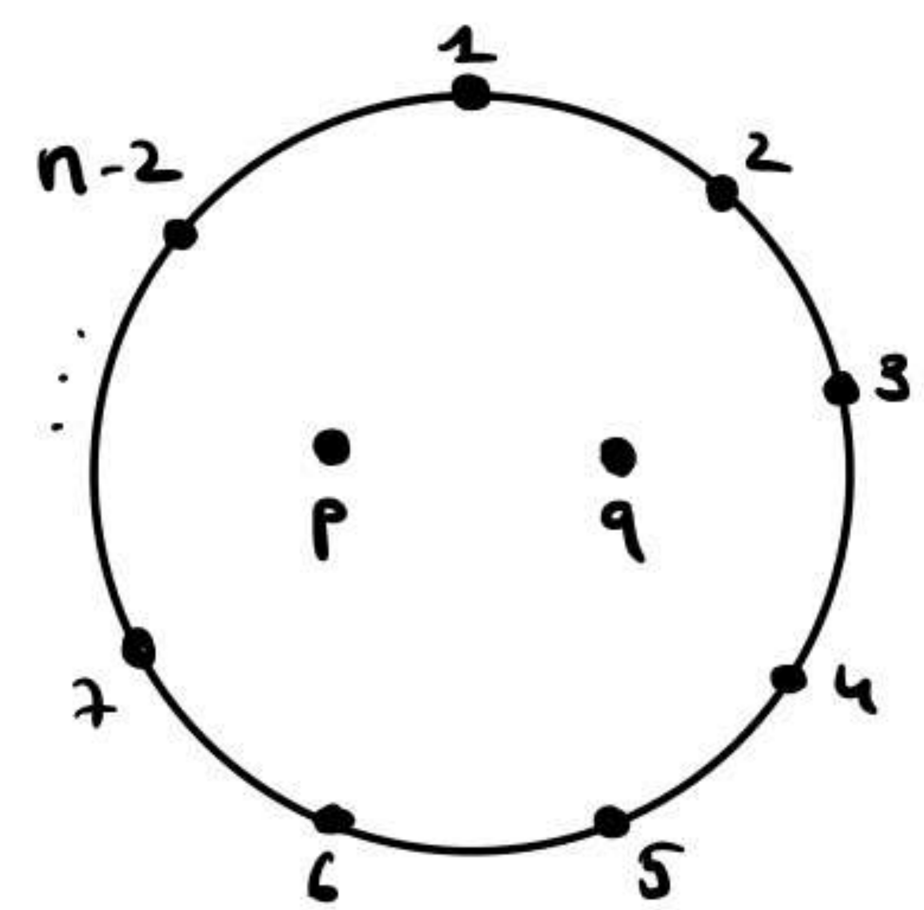
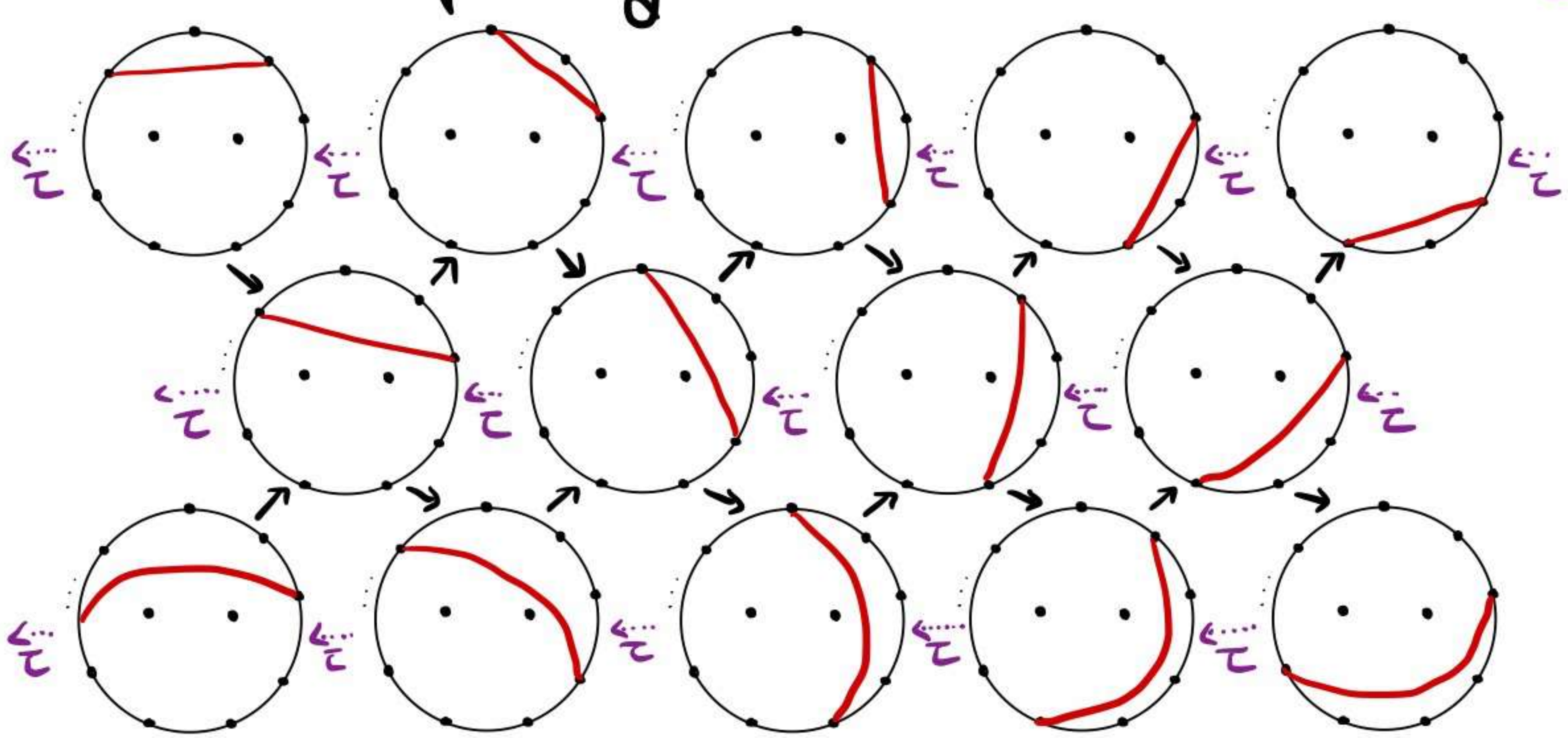


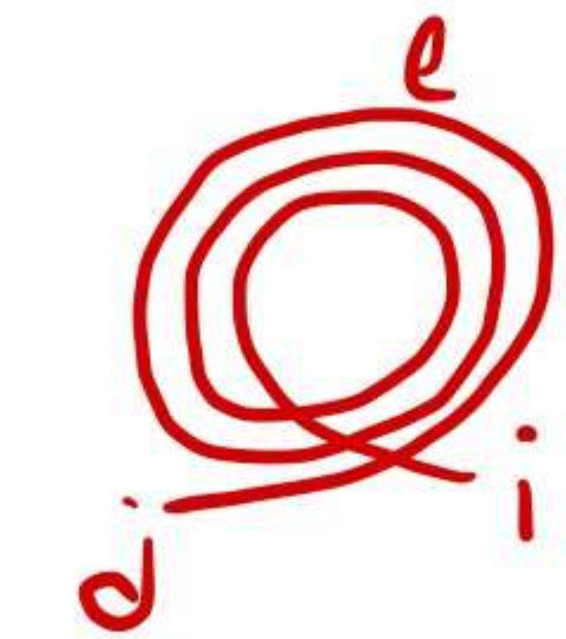
- Arcs corresponding to modules in the tubes: $\mathcal{T}_1$



Proposition [BBGTY, 24]:

$\mathrm{Ind}(\mathcal{T}_1) \longleftrightarrow \{\text{peripheral generalized arcs}\}$

$\gamma_{i,j}^{\ell}$



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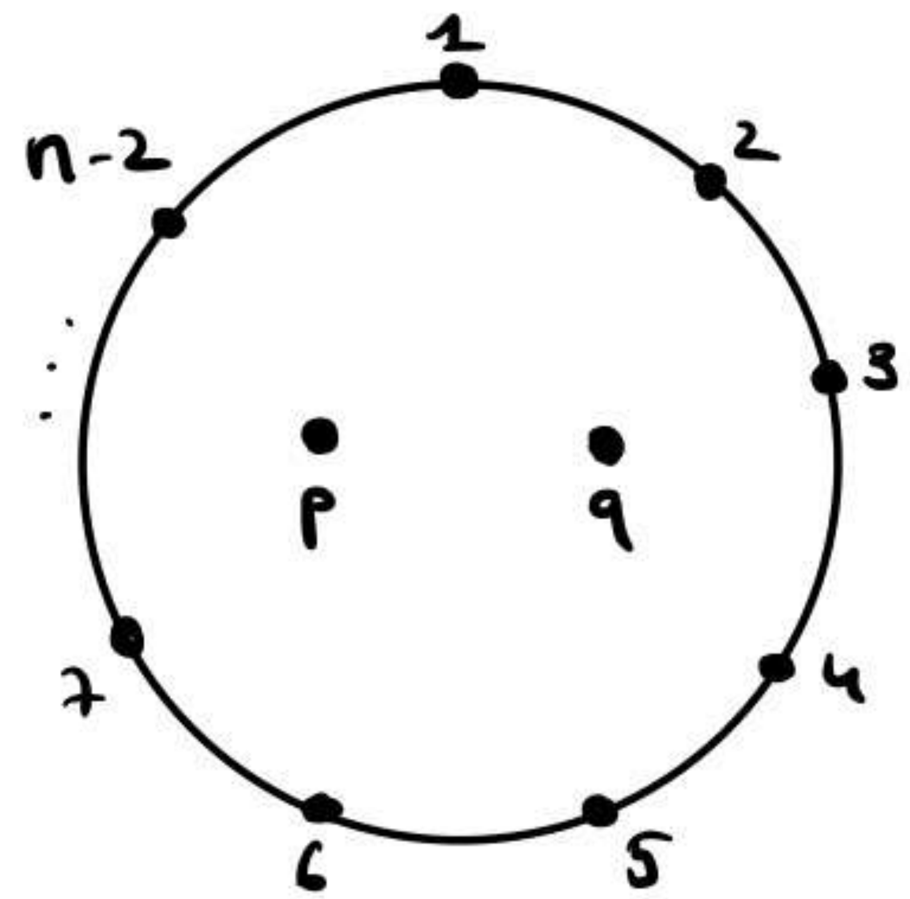

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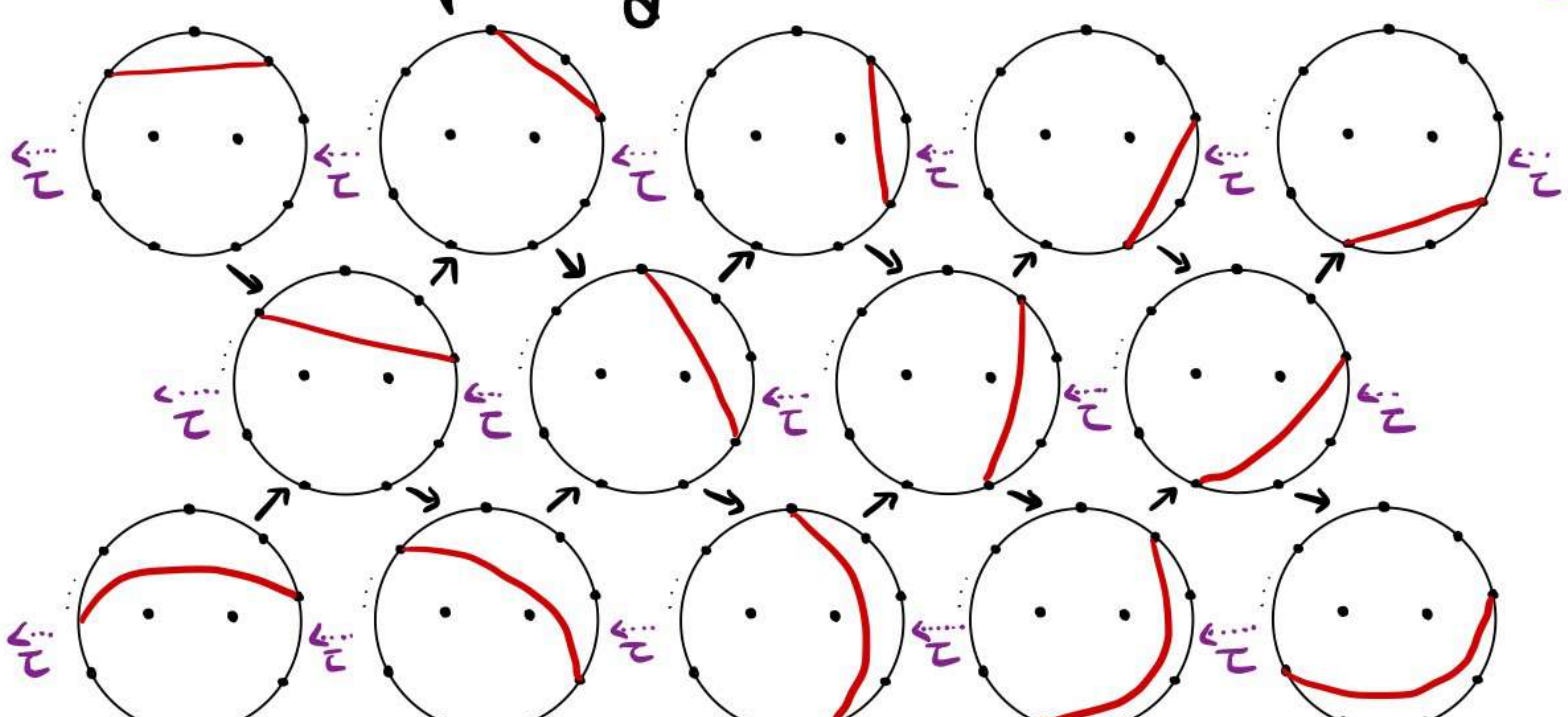

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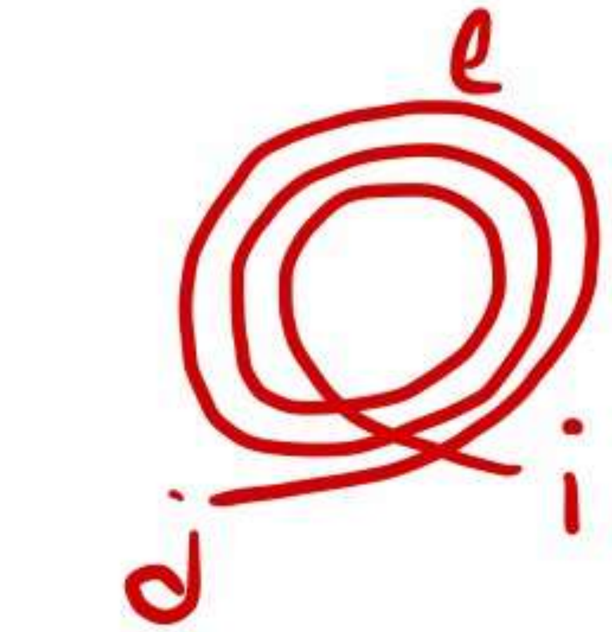
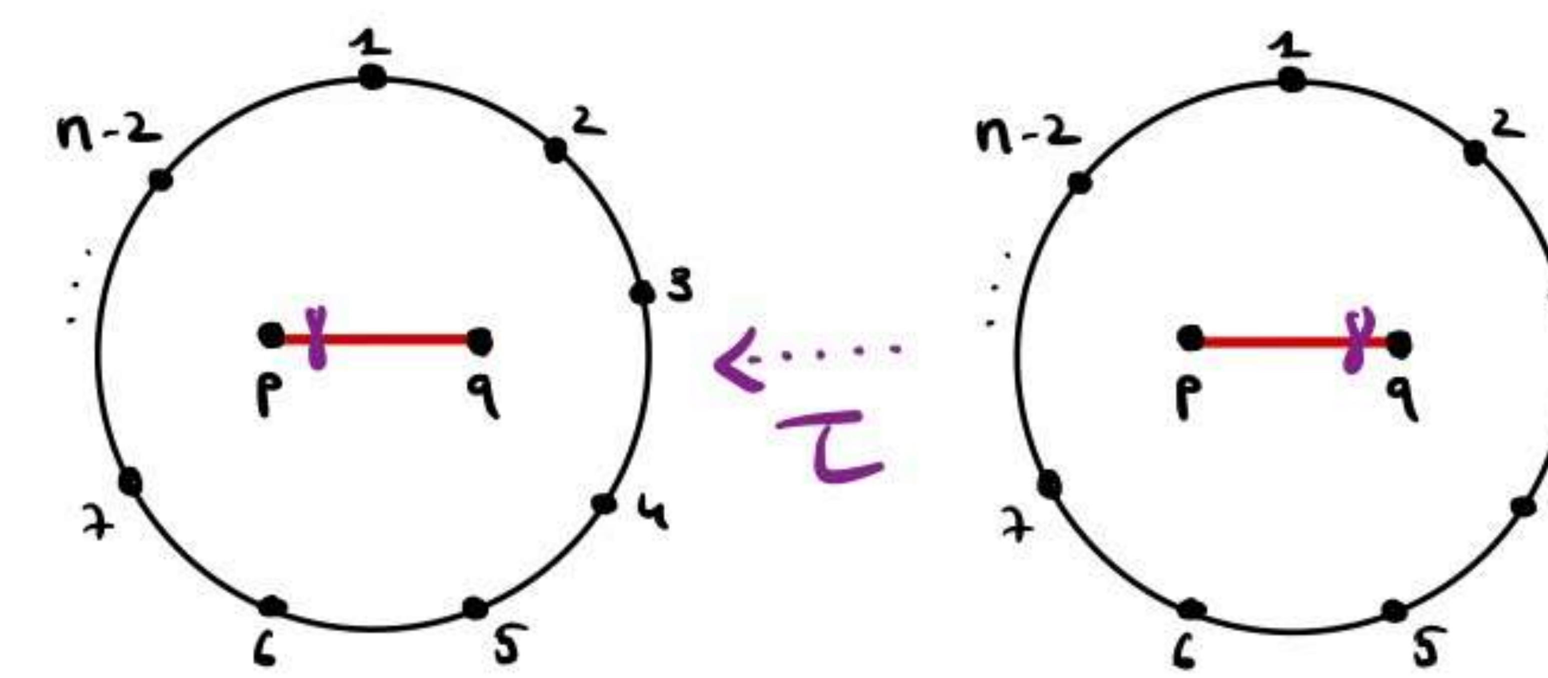


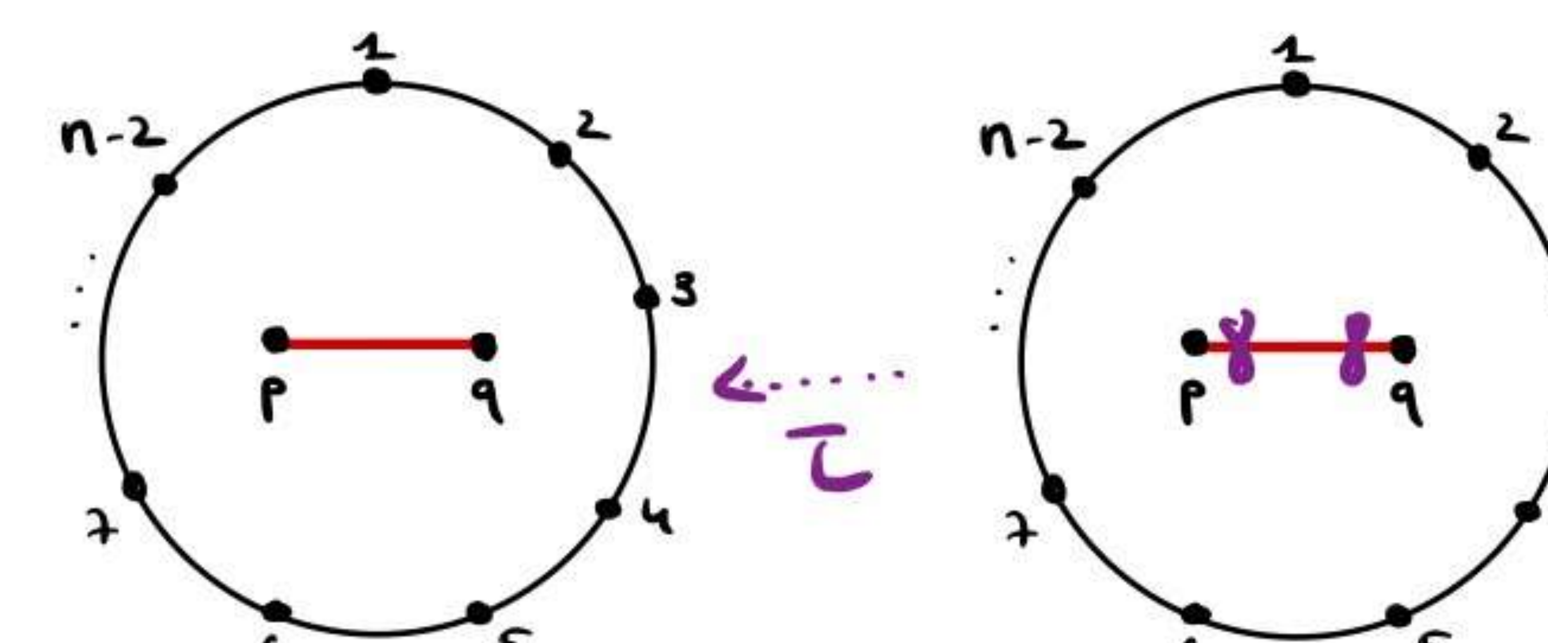
• For tubes $\mathcal{T}_2$ and $\mathcal{T}_3$:

Lemma [BBGTY, 24]: The mouths of the tubes $\mathcal{T}_2$ and $\mathcal{T}_3$ are formed by



$\mathcal{T}_2$

and



$\mathcal{T}_3$

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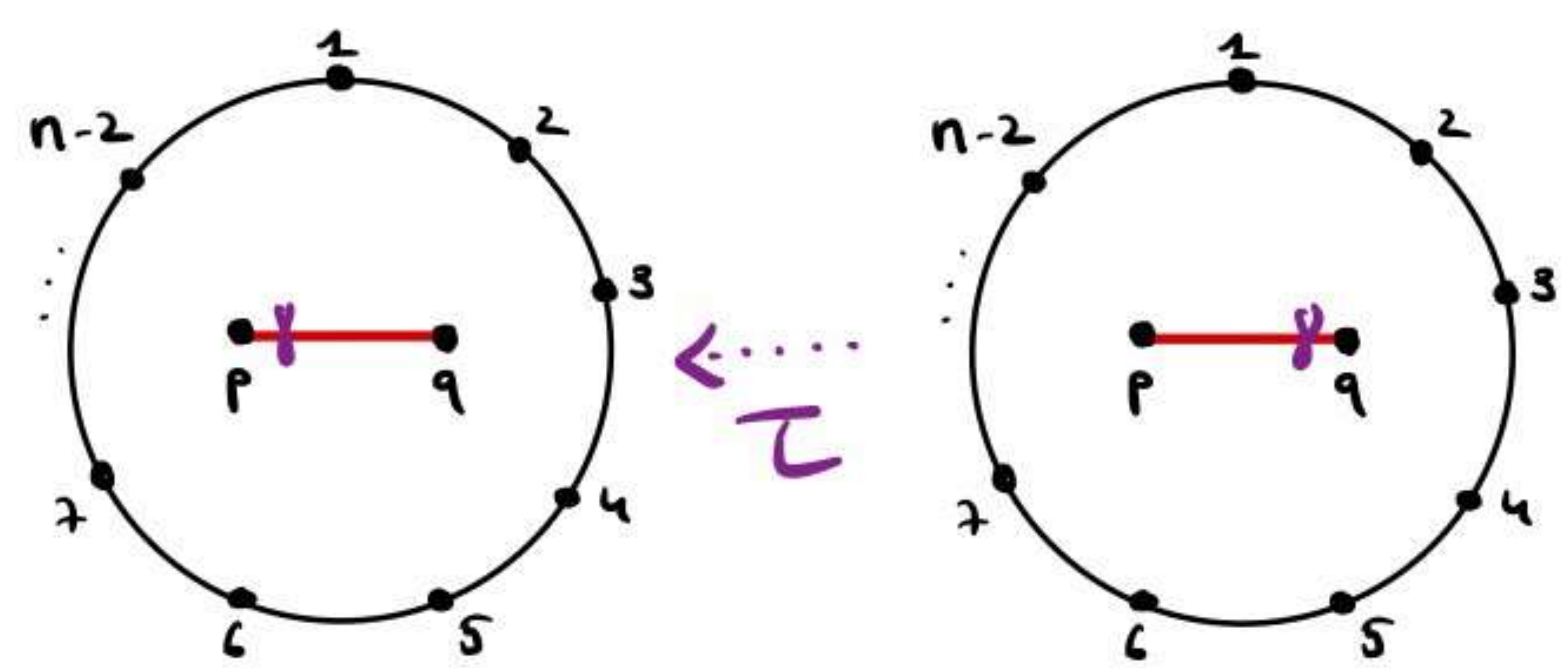

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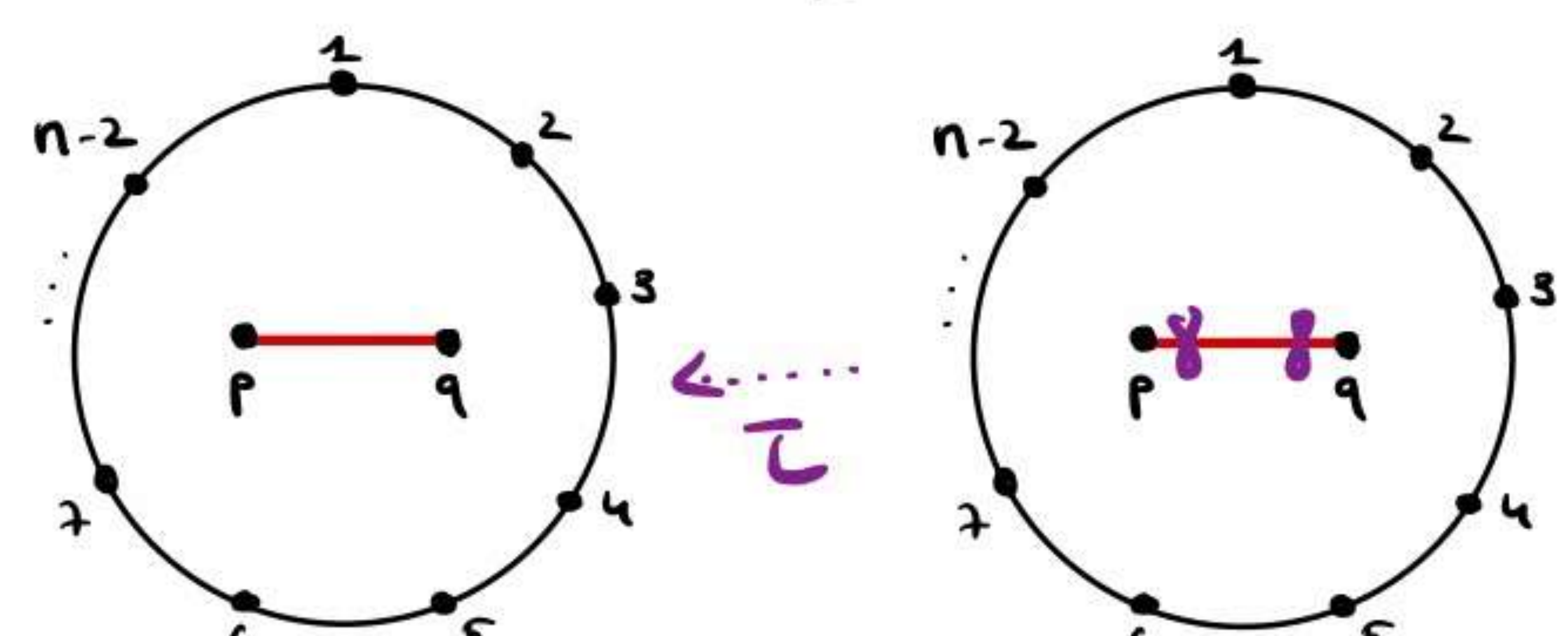
and



$\mathcal{T}_3$

3) Friezes for affine type D:

Cluster character map (CC-map):

$$CC: \mathrm{Ind}(\mathcal{C}_Q) \longrightarrow \mathbb{Q}(x_1, x_2, \dots, x_{n+1})$$

$$M \longmapsto \frac{1}{x_1^{d_1} \cdots x_n^{d_n}} \sum_{\underline{e} \in \mathbb{N}^{Q_0}} \chi(\mathrm{Gr}_{\underline{e}}(M)) \prod_{i \in Q_0} x_i^{\sum_{j \to i} e_j + \sum_{i \to j} (d_j - e_j)}$$

$\underline{\dim} M = \underline{d}$

[Caldero-Chapoton, 06]

For
$$\begin{tikzcd} & E_1 \arrow[dr] & \\ \tau M \arrow[ur] \arrow[dr] & & M \\ & E_2 \arrow[ur] & \end{tikzcd}$$
in the AR-quiver,

$$\boxed{CC(\tau M)CC(M) = 1 + CC(E_1)CC(E_2)}$$

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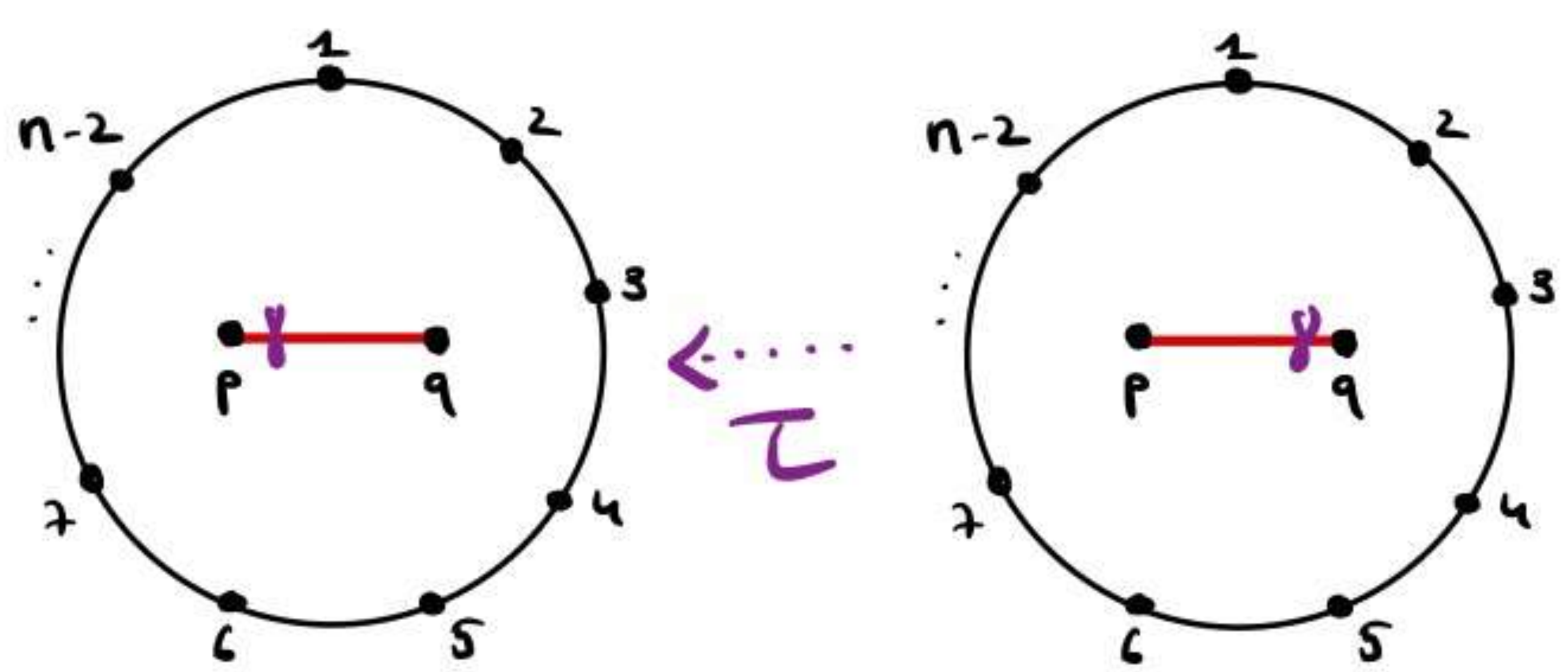
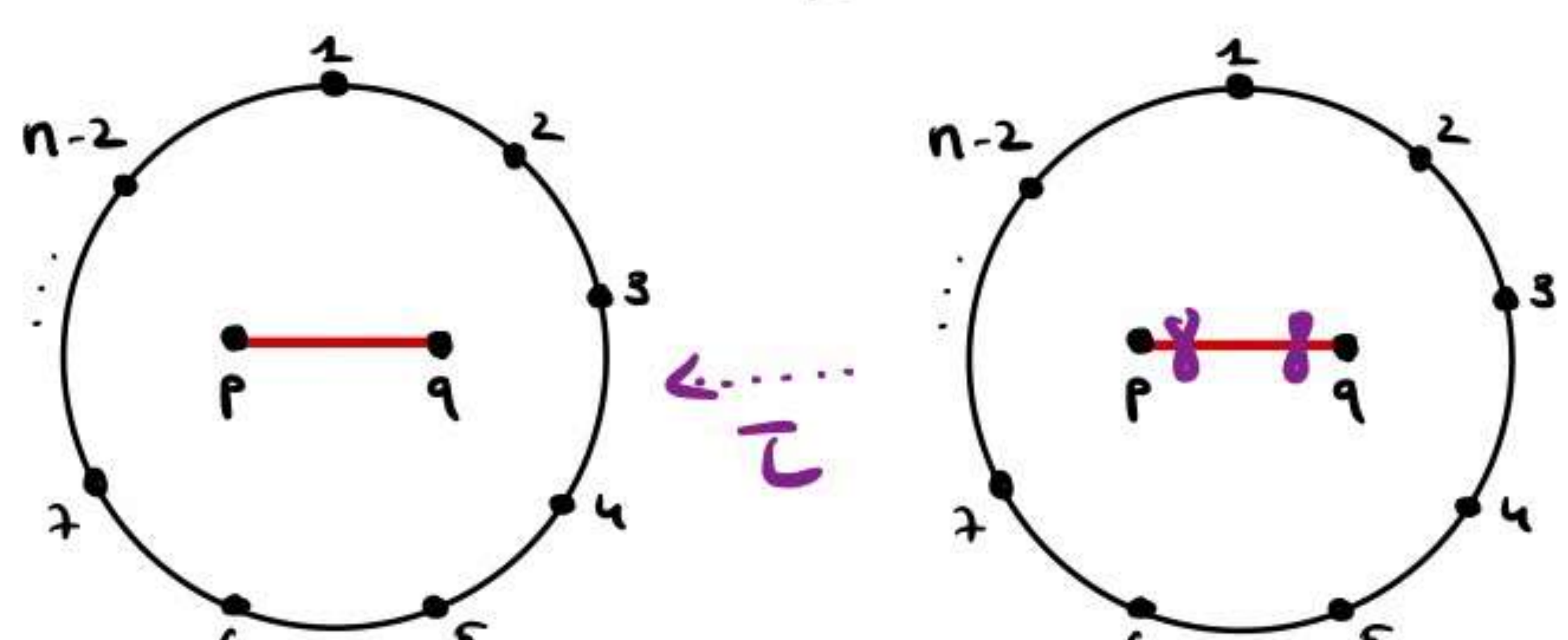


$\mathcal{T}_2$

and



$\mathcal{T}_3$

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in the AR-quiver,

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Consider the specialization ρ of the CC-map to $x_1 = x_2 = \dots = x_{n+1} = \boxed{1}$. Then

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3) Friezes for affine type D:

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$$M \longmapsto \frac{1}{x_1^{d_1}\cdots x_n^{d_n}} \sum_{\underline{e}\in\mathbb{N}^{Q_0}} \chi(Gr_{\underline{e}}(M)) \prod_{i\in Q_0} x_i^{\sum_{j\to i} e_j + \sum_{i\to j}(d_j - e_j)}$$

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[Caldero-Chapoton, 06]

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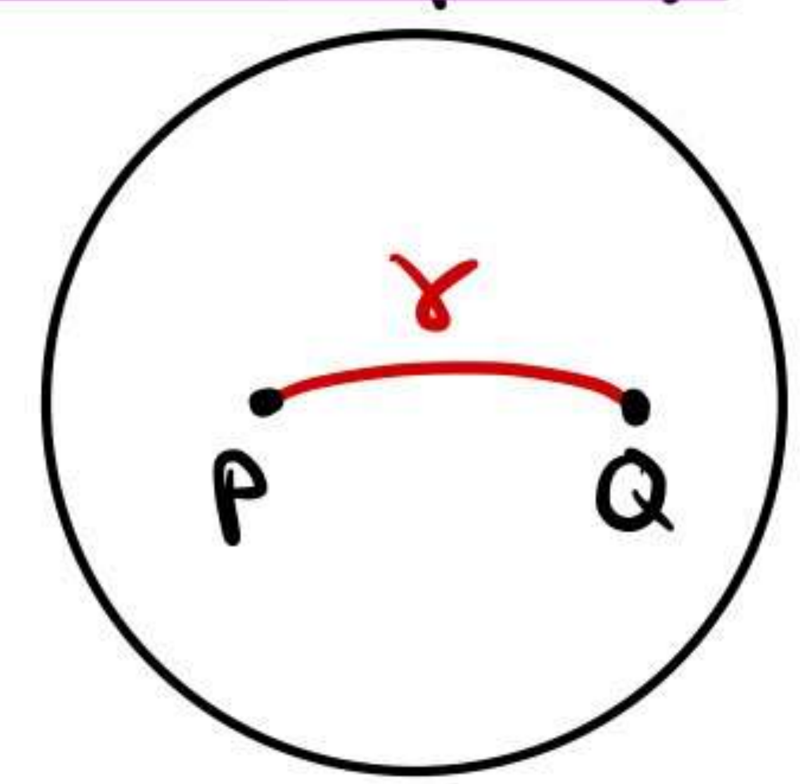
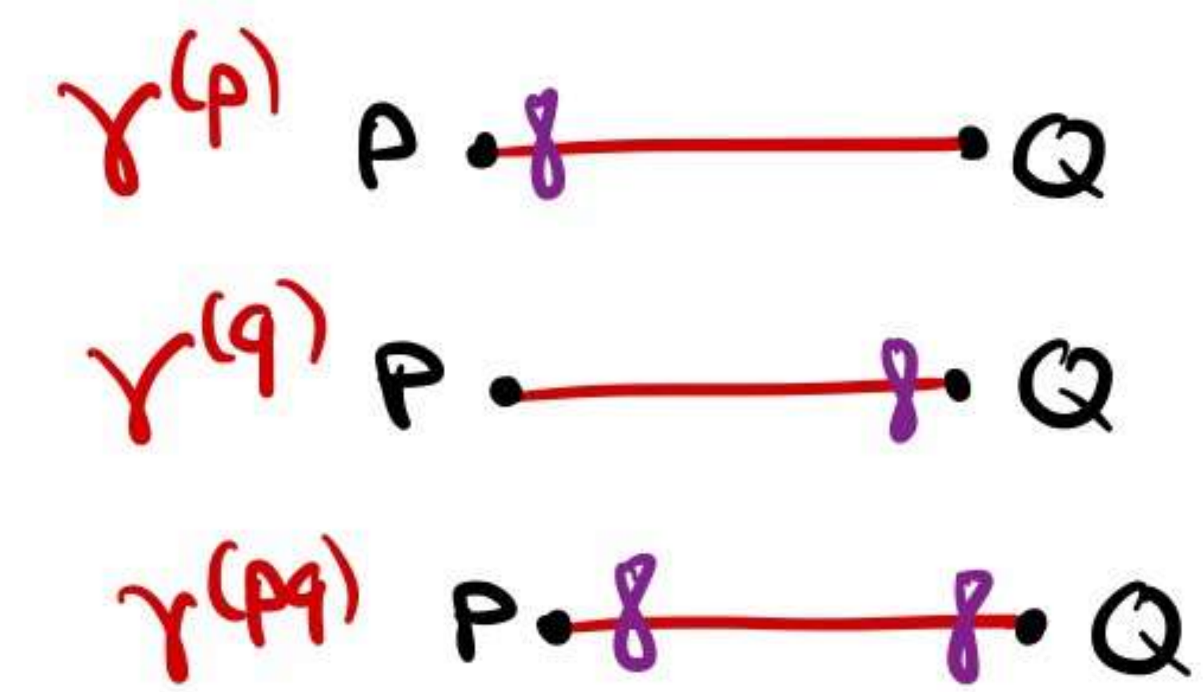


Let $a = \rho(M(\gamma))$



[Musiker-Schiffler-Williams, 11]

$$\rho(M(\gamma^{(p)})) = p \cdot \rho(M(\gamma))$$

- Tube $\mathcal{T}_2$:

the corresponding frieze is

$$\begin{array}{cccc} \cdots & 1 & 1 & 1 \\ \cdots & ap & aq & \cdots \\ \cdots & & a^2pq-1 & \cdots \end{array}$$

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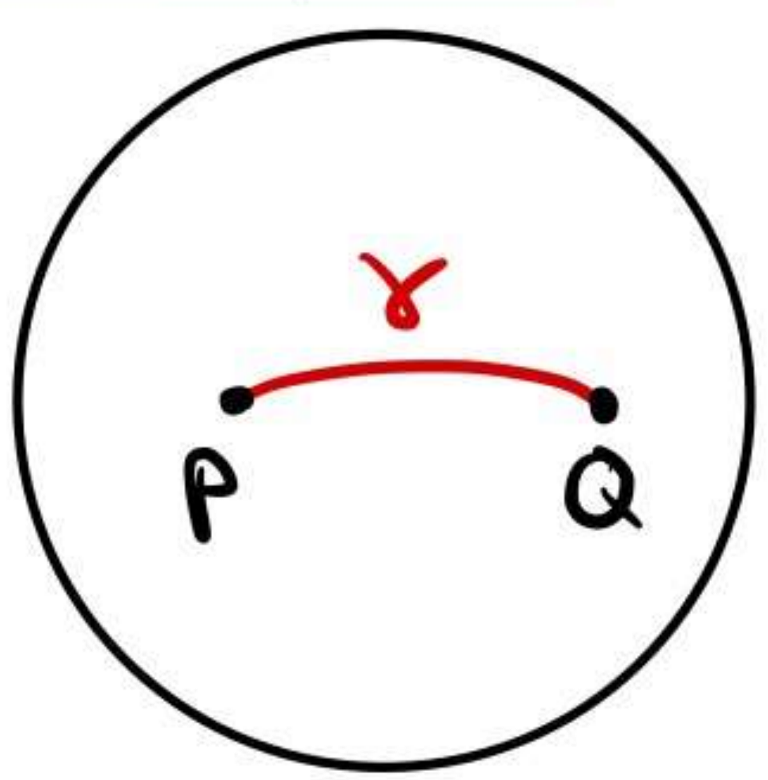

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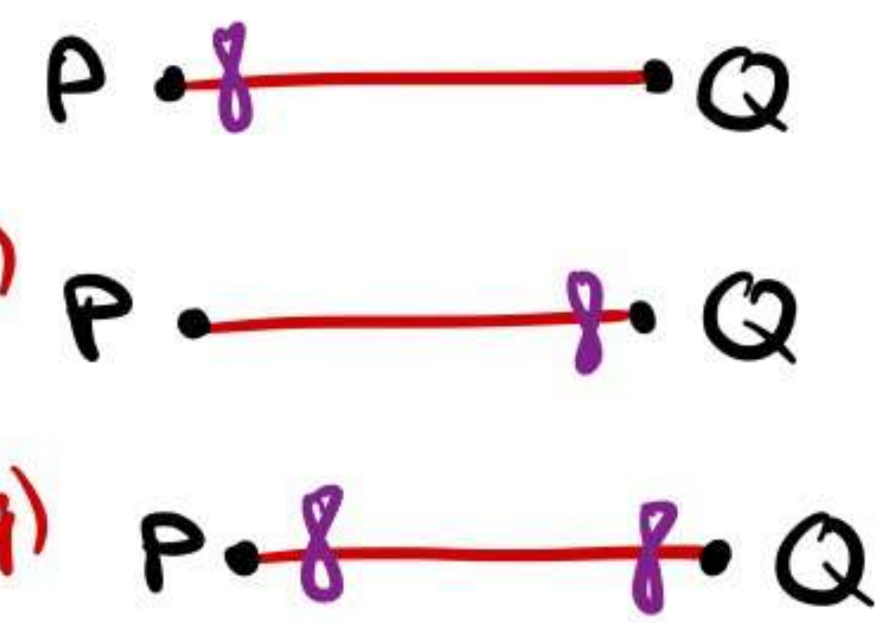

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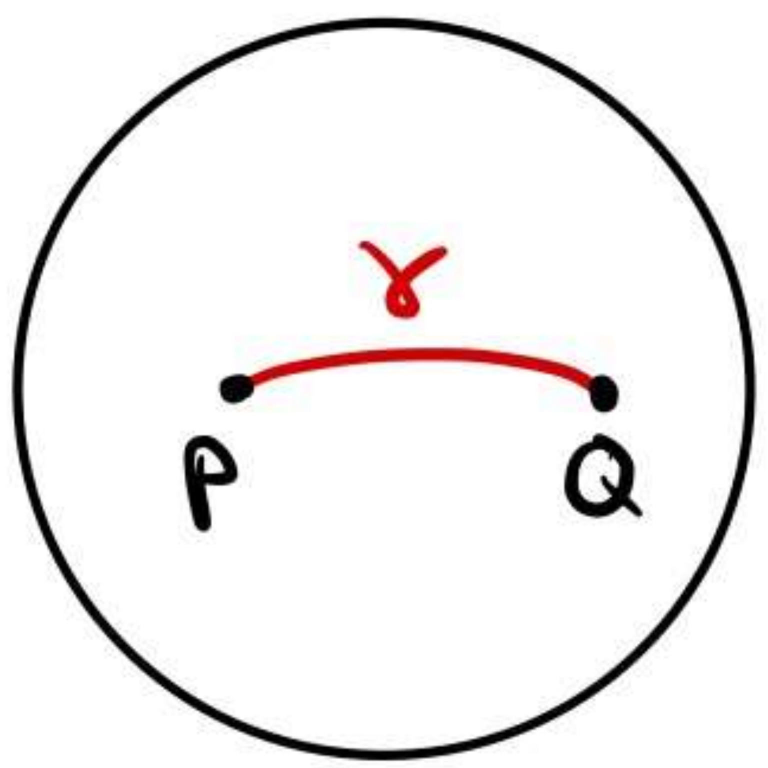

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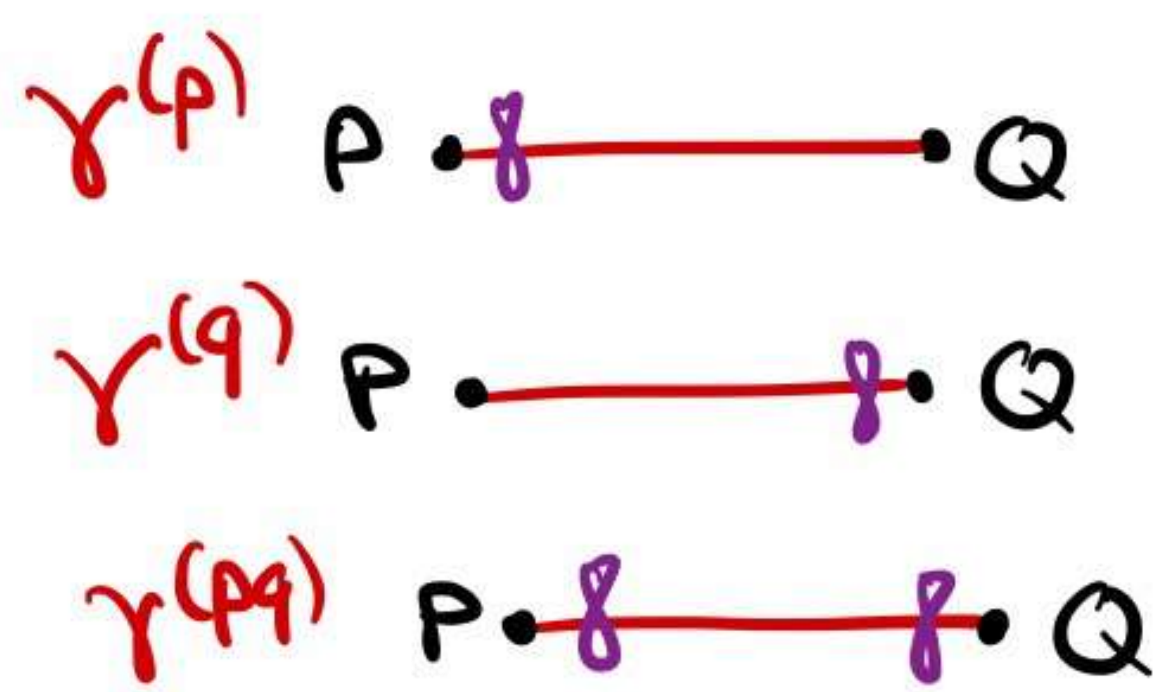

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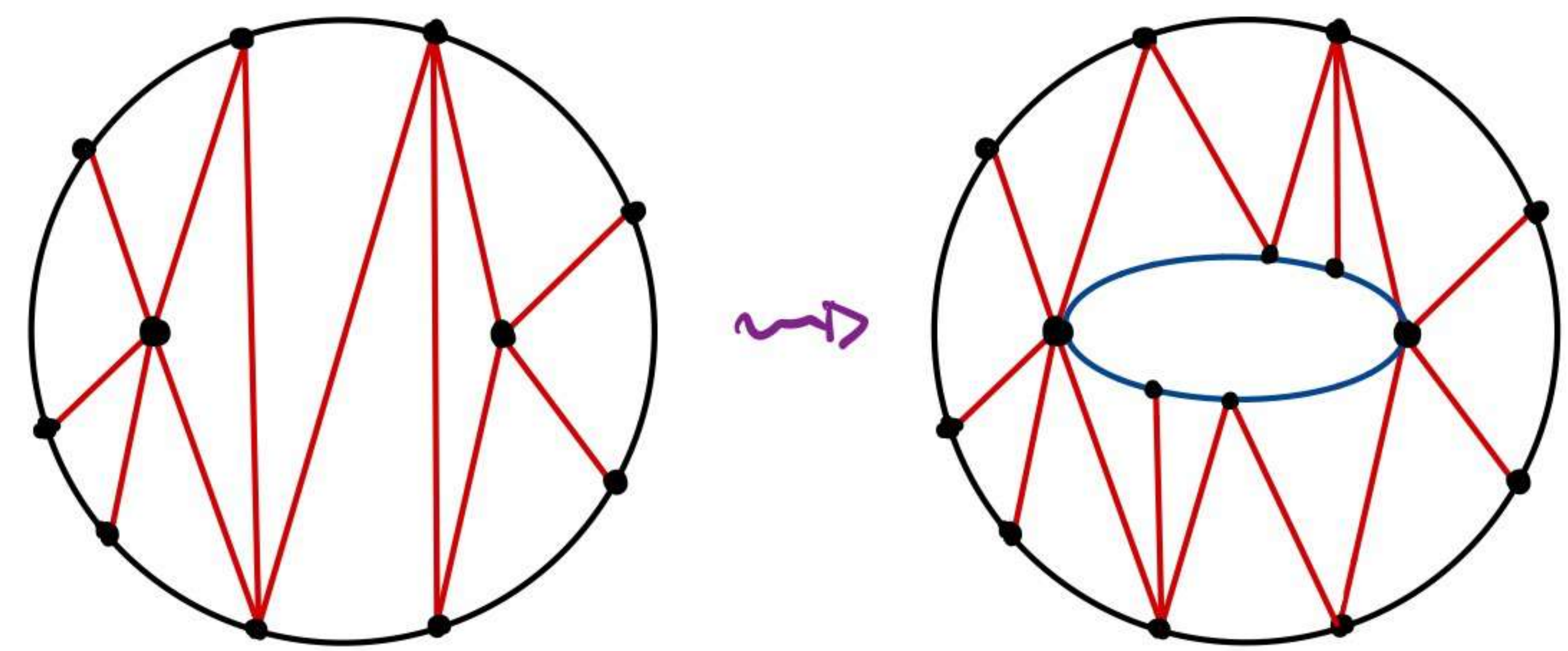

Use triangulations of annulus to show that $\boxed{s = a^2pq - 2}$.

• Tube $\mathcal{T}_3$: 

the corresponding frieze is

$$\begin{array}{ccccc} \cdots & 1 & 1 & 1 & \\ \cdots & a & & apq & \cdots \\ \cdots & & a^2pq-1 & & \cdots \end{array}$$

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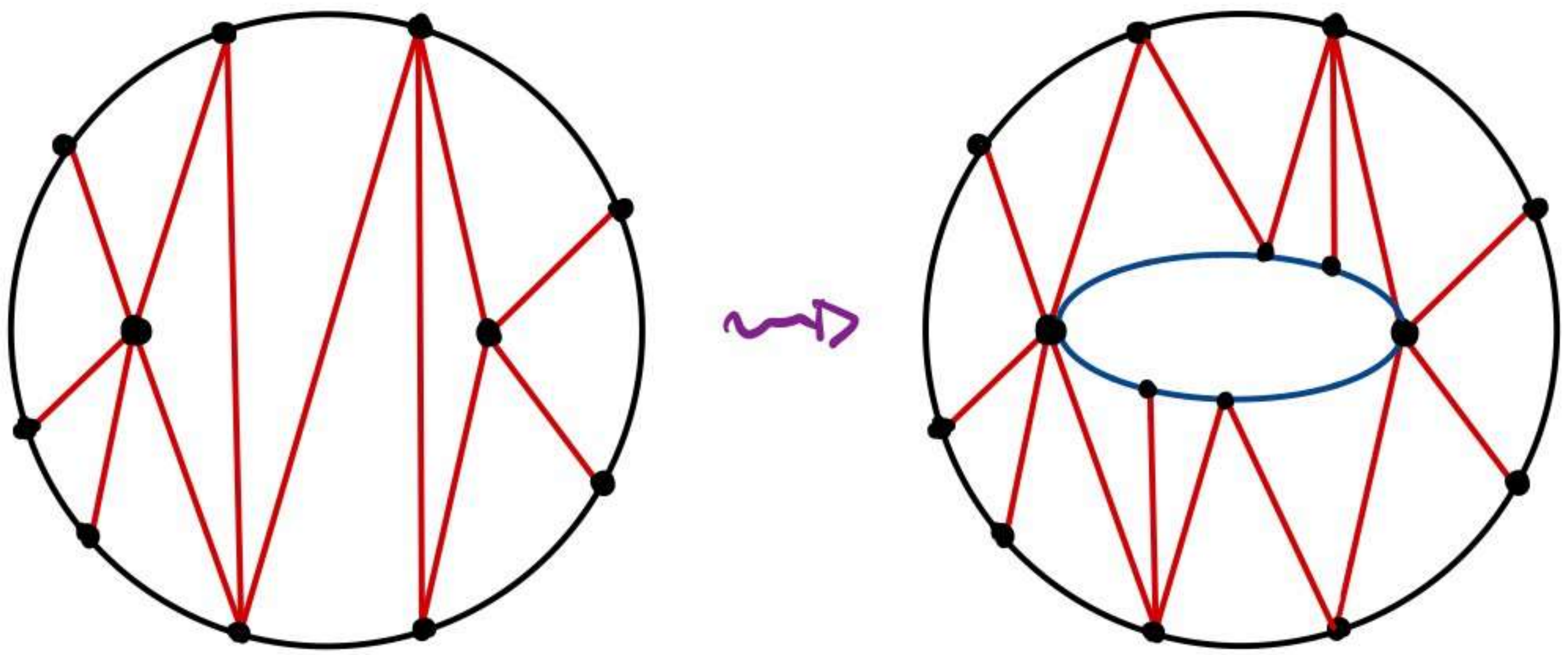

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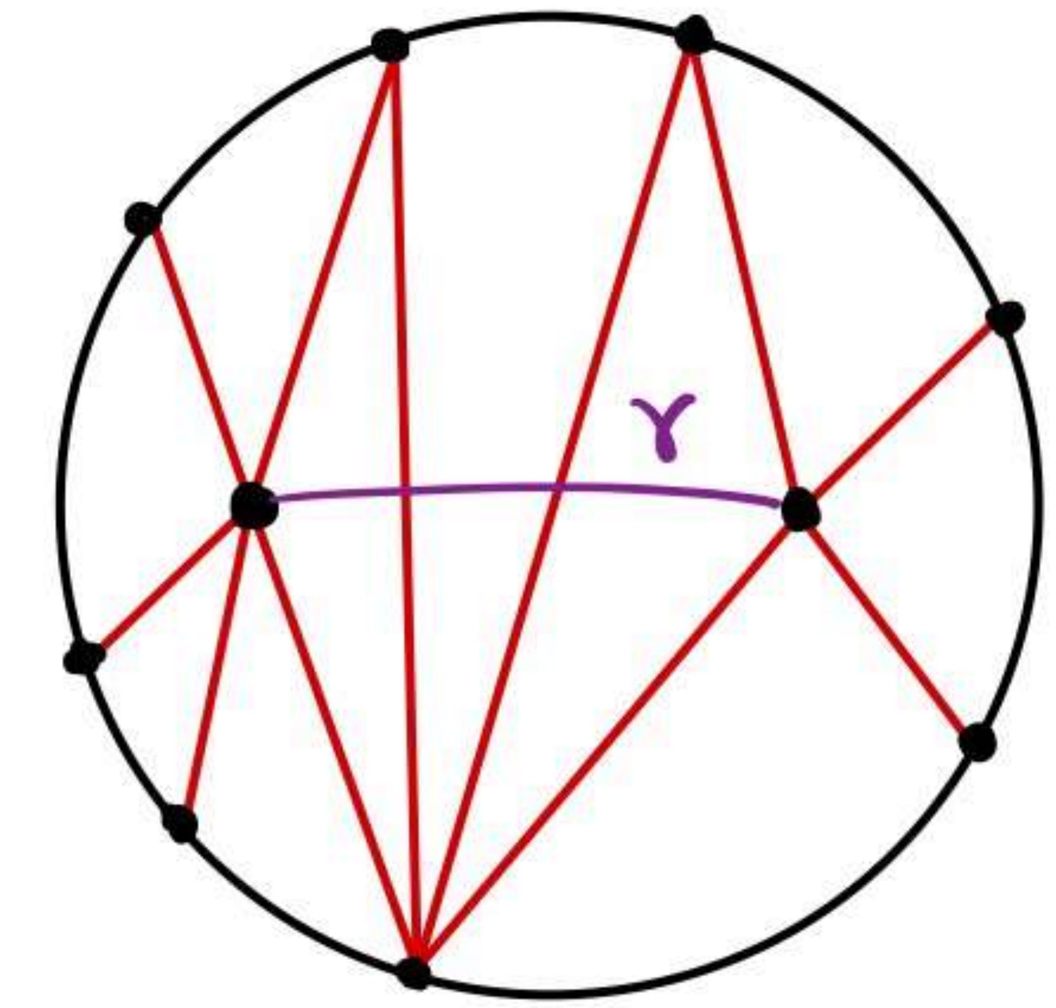
Example:

$Q_\gamma = \longrightarrow$

$a = 3$



$p = 5$

$q = 4$

• Tube $\mathcal{T}_3$: P —— Q   P —γ—γ— Q

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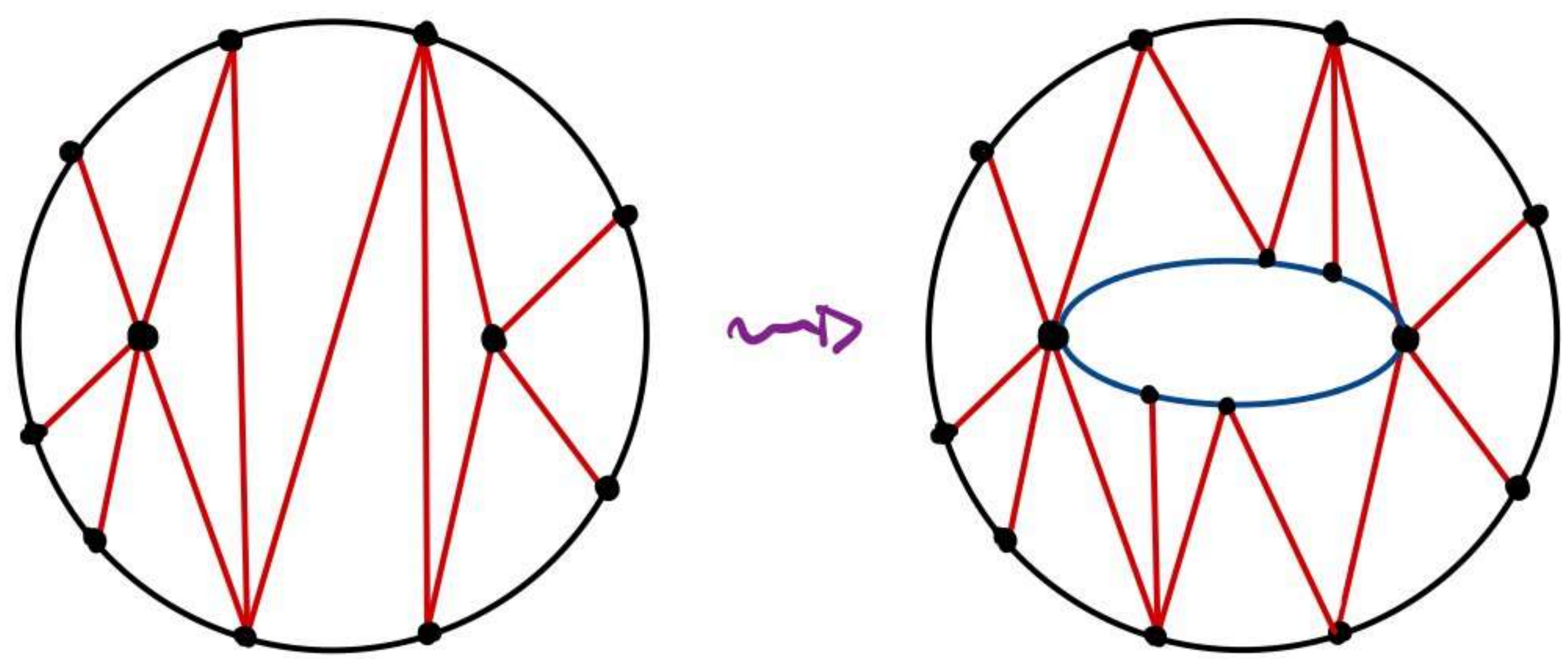

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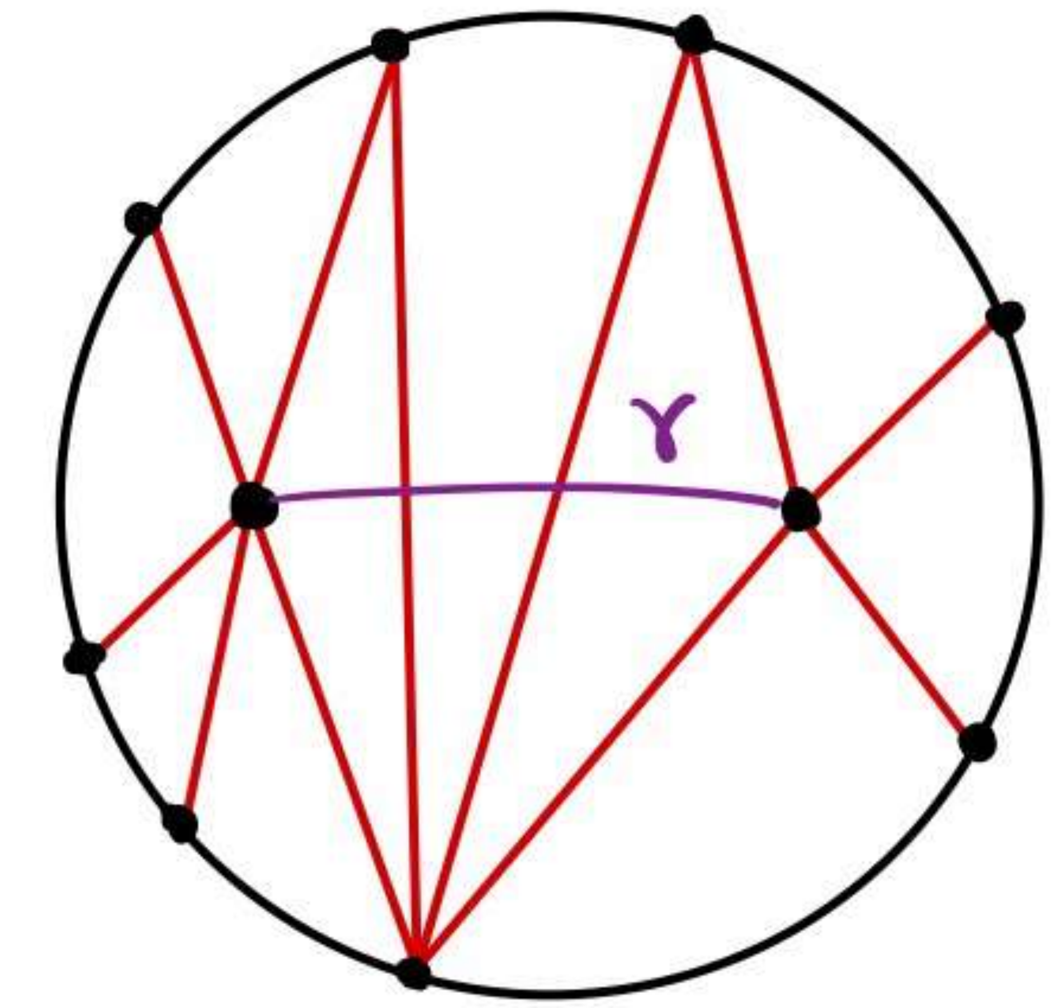

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Tube $\mathcal{T}_1$:

1 1 1 1 1 1 1 1 1 1
2 3 3 2 2 5 2 2 2 3
5 8 5 3 9 9 3 3 5 8
7 13 13 7 13 16 13 4 7 13
18 21 18 30 23 23 17 9 18 21
23 29 29 77 53 33 3 38 23 29
$\boxed{37}$ $\boxed{40}$ $\boxed{124}$ $\boxed{136}$ $\boxed{76}$ $\boxed{43}$ $\boxed{67}$ $\boxed{97}$ 37 40
156 51 171 219 195 99 96 171 156 51
$\boxed{215}$ $\boxed{218}$ $\boxed{302}$ $\boxed{314}$ $\boxed{254}$ $\boxed{221}$ $\boxed{245}$ $\boxed{275}$ 215 218

$S = 254 - 76 = 178 = \boxed{a^2pq - 2}$

Thank you