Representations of G-posets and canonical Brauer induction

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Groups and their actions: algebraic, geometric and combinatorial aspects. Levico Terme, June 3–7, 2024

Overview

1. G-posets and their representations
2. The canonical Brauer induction formula
3. Categorification of the canonical Brauer induction formula

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Example The set of subgroups of G together with the conjugation action of G.

If X is a G-poset, one can form a category $\mathscr{C}(X)$ as follows:

- Objects: the elements of X.
- $\mathrm{Hom}_{\mathscr{C}(X)}(x, y) := \{g \in G \mid x \leq gy\}$.
- Composition: $x \xrightarrow{g} y \xrightarrow{h} z = x \xrightarrow{gh} z$

$(x \leq gy,\ y \leq hz \Rightarrow x \leq gy \leq g(hz) = (gh)z)$.

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Note that $\mathrm{End}_{\mathscr{C}(X)}(x) = G_x^{\mathrm{op}}$, the stabilizer of x in G, with the opposite multiplication $(g, h) \mapsto hg)$. In particular any endomorphism is an isomorphism.

Definition A representation of a G–poset X over k is a functor $F\colon \mathscr{C}(X)^{\mathrm{op}} \to {}_k\mathrm{mod}$. Representations of X over k form an abelian category $\mathcal{P}_k(X)$. Note that for any $g \in G$ and $x \le y$ in X one has commutative diagrams

$$\begin{array}{ccc} gy & \xrightarrow{\;g\;} & y \\ {\scriptstyle 1}\uparrow & & \uparrow{\scriptstyle 1} \\ gx & \xrightarrow{\;g\;} & x \end{array} \qquad \xrightarrow{F} \qquad \begin{array}{ccc} F(gy) & \xleftarrow{\;c_{g,y}\;} & F(y) \\ {\scriptstyle r_{gx}^{gy}}\downarrow & & \downarrow{\scriptstyle r_x^y} \\ F(gx) & \xleftarrow{\;c_{g,x}\;} & F(x) \end{array}$$

Moreover, $F(x)$ is a kG_x-module.

Example Let X be the set of subgroups of G endowed with G-conjugation and let $V \in {}_{kG}\mathrm{Mod}$. One can form the representation $H \mapsto V^H := \{v \in V \mid hv = v \text{ for all } h \in H\}$. This defines a functor

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The restriction maps are inclusions and the conjugation map $c_{g,H}$ is the application of g on V^H.

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This is also the category algebra $k\mathscr{C}(X)^{\mathrm{op}}$. If X is finite, $A(X)$ has the identity element

$$1_{A(X)} = \sum_{x \in X} e_x \,,$$

where $e_x = (x, 1, x) = \mathrm{id}_x$.

Proposition *If X is a finite G-poset then one has a category equivalence*

$$\mathcal{P}_k(X) \cong {}_{A_k(X)}\mathrm{mod}\,.$$

The simple $A_k(X)$-modules are parametrized by G-orbits of pairs $(x, [V])$, where $x \in G$ and V is a simple kG_x-module.

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Proposition (B.-Monteiro 2024) *One can explicitly determine the central idempotents of $A_k(X)$ in terms of central idempotents of the various group algebras kG_x.*

2. The canonical Brauer induction formula

In this section, $k = \mathbb{C}$.
$R(G) :=$ ring of virtual characters of $G =$ Grothendieck ring of ${}_{\mathbb{C}G}\mathrm{mod}$.
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Theorem (Brauer 1947) *For every $\chi \in R(G)$ there exist $H_i \leq G$, $\varphi_i \in \hat{H}_i$, $n_i \in \mathbb{Z}$, $i = 1, \ldots, r$, such that*

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Consider the set

$$\mathcal{M}_G := \{(H, \varphi) \mid H \leq G, \varphi \in \hat{H}\}\,.$$

It is a G-poset via $(K, \psi) \leq (H, \varphi) :\iff K \leq H$ and $\psi = \varphi|_K$, together with the G-conjugation action $(g, (H, \varphi)) \mapsto {}^{g}(H, \varphi) = ({}^{g}H, {}^{g}\varphi)$.

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Also, the diagram

$$\begin{array}{ccc} R_+(G) & \xrightarrow{b_G} & R(G) \\ {\scriptstyle \mathrm{res}^G_H}\downarrow & & \downarrow{\scriptstyle \mathrm{res}^G_H} \\ R_+(H) & \xrightarrow{b_H} & R(H) \end{array}$$

commutes.

Definition (B. 1990) A canonical Brauer induction formula is a family of maps $a_G\colon R(G) \to R_+(G)$, one for each finite group G, such that $b_G \circ a_G = \mathrm{id}_{R(G)}$ and a_G commutes with restrictions to subgroups.

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• B. 1990: Canonical induction formulas are uniquely determined up to a normalization. The most obvious normalization leads to **the** canonical Brauer induction formula, explicitly given by

$$a_G(\chi) = \sum_{\substack{(H_0,\varphi_0)<\cdots<(H_n,\varphi_n)\\ \mathrm{mod}\, G}} (-1)^n (\chi|_{H_n}, \varphi_n)[H_0, \varphi_0]_G\,.$$

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Thus, if χ is afforded by $V \in {}_{\mathbb{C}G}\mathrm{mod}$ then

$$\chi = \sum_{\substack{(H_0,\varphi_0)<\cdots<(H_n,\varphi_n) \\ \mathrm{mod}\, G}} (-1)^n \mathrm{ind}_{H_0}^G [V^{(H_n,\varphi_n)}] \,,$$

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- Symonds 1991: geometric interpretation of this formula.

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• Objects: Pairs $(M, \mathscr{L})$ with $M \in {}_{\mathbb{C}G}\mathrm{mod}$ and $\mathscr{L} = \{L_1, \ldots, L_n\}$ a set of 1-dimensional $\mathbb{C}$-subspaces of M that are permuted by G and satisfy $M = L_1 \oplus \cdots \oplus L_n$.

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Every L_i has a stabilizing pair $(H_i, \varphi_i) \in \mathcal{M}_G$. For $(H, \varphi) \in \mathcal{M}_G$ set

$$M((H,\varphi)) := \bigoplus_{\substack{L_i \in \mathscr{L} \\ (H_i,\varphi_i)=(H,\varphi)}} L_i \quad \text{and} \quad M^{((H,\varphi))} := \bigoplus_{\substack{L_i \in \mathscr{L} \\ (H_i,\varphi_i)\geq(H,\varphi)}} L_i \,.$$

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• $\mathrm{Hom}_{{}_{\mathbb{C}G}\mathsf{mon}}(M, N)$ is the set of $f \in \mathrm{Hom}_{\mathbb{C}G}(M, N)$ satisfying

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${}_{\mathbb{C}G}\mathrm{mon}$ is a $\mathbb{C}$-linear additive category, but not abelian.

Proposition (B. 2001) *Every indecomposable object in* ${}_{\mathbb{C}G}\mathrm{mon}$ *is of the form* $\mathrm{Ind}_H^G(\mathbb{C}_\varphi) = \mathbb{C}G \otimes_{\mathbb{C}H} \mathbb{C}_\varphi$ *for some* $(H, \varphi) \in \mathcal{M}_G$*, uniquely determined up to conjugation, and the Grothendieck group of* ${}_{\mathbb{C}G}\mathrm{mon}$ *is* $R_+(G)$*.*

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Definition The functors $\mathcal{I} \colon {}_{\mathbb{C}G}\mathrm{mod} \to \mathcal{P}(\mathcal{M}_G)$ and $\mathcal{J} \colon {}_{\mathbb{C}G}\mathrm{mon} \to \mathcal{P}(\mathcal{M}_G)$ are defined by

$$\mathcal{I}(V) = \left(V^{(H,\varphi)}\right)_{(H,\varphi)\in\mathcal{M}_G} \quad \text{and} \quad \mathcal{J}(M) := \left(M^{((H,\varphi))}\right)_{(H,\varphi)\in\mathcal{M}_G}.$$

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Proposition (B. 2001) *$\mathcal{I}$ and $\mathcal{J}$ are fully faithful embeddings of ${}_{\mathbb{C}G}$mod and ${}_{\mathbb{C}G}$mon into the full subcategory $\mathcal{P}'(\mathcal{M}_G)$ of $\mathcal{P}(\mathcal{M}_G)$ consisting of those functors F, such that $h \in H$ acts on $F(H, \varphi)$ via multiplication with $\varphi(h)$.*

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Moreover, if $\mathcal{J}(M_*)$ is this projective resolution of $\mathcal{I}(V)$ then $a_G([V]) = \sum_{i\geq 0}(-1)^i[M_i]$ in $R_+(G)$. In particular, one has a commutative diagram

$$\begin{array}{ccc} {}_{\mathbb{C}G}\mathrm{mod} & \rightarrow & K^b({}_{\mathbb{C}G}\mathrm{mon}) \\ {\scriptstyle [-]}\downarrow & & \downarrow{\scriptstyle [-]} \\ R(G) & \xrightarrow{a_G} & R_+(G) \end{array}$$

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Remark For given $V \in {}_{\mathbb{C}G}\mathrm{mod}$ one can find an M_* of length $\leq$ longest strictly ascending chain in the set of subspaces $V^{(H,\varphi)} \neq 0$.

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Set

$$\varepsilon := \sum_{(H,\varphi)\in\mathcal{M}} \frac{1}{|H|} \sum_{h\in H} \varphi(h^{-1}) \cdot \big((H,\varphi), h, (H,\varphi)\big) \in A(\mathcal{M}_G)\,.$$

Then ε is an idempotent and $\mathcal{P}'(\mathcal{M}_G)$ corresponds under the equivalence $\mathcal{P}(\mathcal{M}_G) \cong {}_{A(\mathcal{M}_G)}\text{mod}$ to the full subcategory of ${}_{A(\mathcal{M}_G)}\text{mod}$ consisting of those modules on which ε acts as identity.

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$$\varepsilon := \sum_{(H,\varphi)\in\mathcal{M}} \frac{1}{|H|} \sum_{h\in H} \varphi(h^{-1}) \cdot \big((H,\varphi), h, (H,\varphi)\big) \in A(\mathcal{M}_G)\,.$$

Then ε is an idempotent and $\mathcal{P}'(\mathcal{M}_G)$ corresponds under the equivalence $\mathcal{P}(\mathcal{M}_G) \cong {}_{A(\mathcal{M}_G)}\mathrm{mod}$ to the full subcategory of ${}_{A(\mathcal{M}_G)}\mathrm{mod}$ consisting of those modules on which ε acts as identity.

Proposition (B.-Monteiro 2024) *One has $A(\mathcal{M}_G)\varepsilon \subseteq \varepsilon A(\mathcal{M}_G)$. In particular, $A(\mathcal{M}_G)\varepsilon = \varepsilon A(\mathcal{M}_G)\varepsilon$ and $a\varepsilon = \varepsilon a \varepsilon$ for all $a \in A(\mathcal{M}_G)$.*

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Example For $F \in \mathcal{P}(\mathcal{M}_G)$ and fixed $(H,\varphi) \in \mathcal{M}_G$ consider the functor

$$F \mapsto F(H,\varphi)/\sum_{(H,\varphi)\le(H',\varphi')} r_{(H,\varphi)}^{(H',\varphi')}(F(H',\varphi')) \in {}_{\mathbb{C}G_{(H,\varphi)}}\mathrm{mod}\,.$$

Thank You