THE STABLE REPRESENTATIONS OF THE GENERAL LINEAR GROUP (GL_n) OVER FINITE LOCAL RINGS

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REPRESENTATION OF FINITE GROUP (G) OVER $\mathbb{C}$

IS A GROUP HOMOMORPHISM $\rho : G \to GL_n(\mathbb{C})$

CHARACTER $\Rightarrow$ $\chi : G \to \mathbb{C}$ $\Rightarrow$ $\chi(g) = Tr(\rho(g))$

LET $Irr(G)$ BE THE SET OF IRREDUCIBLE CHARACTER OF G

OVER $\mathbb{C}$.

GOAL: CONSTRUCT IRREDUCIBLE REPRESENTATIONS OF CERTAIN GROUPS

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② WHAT GROUPS?

$\mathcal{O} \Rightarrow$ PRINCIPAL IDEAL DOMAIN WITH UNIQUE MAXIMAL IDEAL $M = \pi\mathcal{O}$

↑ GENERATOR

$\mathbb{F}_q \cong \mathcal{O}/M$ HAS CHARACTERISTIC p, p PRIME

$\mathcal{O}_r = \mathcal{O}/M^r$

EXAMPLES TO KEEP IN MIND:

	①	②
$\mathcal{O}$	$\mathbb{F}_p[[t]]$	$\mathbb{Z}_p$
$\mathcal{O}_r$	$\mathbb{F}_p[t]/t^r$	$\mathbb{Z}_p/p^r \simeq \mathbb{Z}/p^r\mathbb{Z}$
π	$\pi = t$	$\pi = p$

$\Rightarrow$ FINITE LOCAL RINGS

WE CAN THUS FORM THE GROUP $GL_n(\mathcal{O}_r) =: G_r$

EXAMPLES: $GL_n(\mathbb{Z}/p^r\mathbb{Z})$ AND $GL_n(\mathbb{F}_p[t]/t^r)$

A LITTLE BIT OF HISTORY ABOUT THE REPRESENTATIONS OF $GL_m(\mathcal{O}_r)$

① FOR $GL_m(\mathcal{O}_1) = GL_m(\mathbb{F}_q)$, $\mathrm{Irr}(GL_m(\mathbb{F}_q))$ DETERMINED IN A CLASSICAL PAPER BY GREEN (1955)

CAN ALSO BE CONSTRUCTED VIA DELIGNE-LUSZTIG THEORY.

② $\mathrm{Irr}(GL_m(\mathcal{O}_2))$ WERE DETERMINED BY SINGLA (2010)

MOREOVER, $\mathbb{C}GL_m(\mathcal{O}_2) \cong \mathbb{C}GL_m(\mathcal{O}_2')$, $\mathcal{O}_1 \cong \mathcal{O}_1'$

③ IN GENERAL FOR $GL_m(\mathcal{O}_r)$ WITH $m \geq 2, r \geq 2$ ONLY CERTAIN CLASSES OF IRREDUCIBLE REPRESENTATIONS ARE KNOWN:

$Gl_n(\mathcal{O}_r)$

(3.1) IRREDUCIBLE REGULAR REPRESENTATIONS WHEN r IS EVEN (SHINTANI, 1968) AND (HILL, 1995)

(3.2) CONSTRUCTION OF IRR. STRONGLY SEMISIMPLE REPRESENTATIONS (HILL, 1995)

(3.3) TWO INDEPENDENT GENERAL CONSTRUCTION OF IRR. REGULAR REPRESENTATIONS FOR:

- $p \neq 2$, [KRAKOVSKI, ONN, SINGLA, 2018]
- FOR ALL p, [STASINSKI, STEVENS, 2017]

(3.4) STABLE REPRESENTATIONS [M, 2023]

(3) CLIFFORD THEORY / SETUP OF OUR GROUPS

(3.1) NORMAL SUBGROUPS OF $GL_n(\mathcal{O}_r)$

NOTATION: $G_r = GL_n(\mathcal{O}_r)$; $\mathfrak{g}_r := M_n(\mathcal{O}_r)$; WE FIX r

FOR EACH $1 \leq i \leq r$, $P_i : G_r \to G_i$

"TAKE A MATRIX AND REDUCE IT mod π^i"

$K^i := \operatorname{Ker} P_i = I + \pi^i\left(M_n(\mathcal{O}_r)\right)$ WITH $I :=$ IDENTITY MATRIX.

$I + \pi^i X$

WE THUS HAVE A DESCENDING CHAIN OF NORMAL SUBGROUPS:

$$G_r \supset K^1 \supset \cdots \supset K^r = \{I\}.$$

LEMMA: FOR $i \geq r/2$ THEN K^i IS ABELIAN, IN FACT

$$K^i \cong M_n(\mathcal{O}_{r-i}), \quad i \geq r/2$$

$$I + \pi^i X \mapsto X \bmod \pi^{r-i}$$

REMARK: IF $r = 2\ell$ OR $r = 2\ell - 1$ THEN K^ℓ IS THE MAXIMAL NORMAL ABELIAN GROUP AMONG THE K^i

EXAMPLE: $r = 5 \Rightarrow \ell = 3$; $r = 4 \Rightarrow \ell = 2$

3.2 CLIFFORD THEORY:

NOTATION: G FINITE GROUP, $N \trianglelefteq G$, $\psi \in \mathrm{Irr}(N)$

$G(\psi) = \{ g \in G \mid \psi^g = \psi \} \Rightarrow$ STABILIZER OF ψ IN G

$\psi^g(n) := \psi(g n g^{-1})$ WITH $n \in N$, AND $g \in G$

$\mathrm{Irr}(G \mid \psi) = \{ \chi \in \mathrm{Irr}(G) \mid \langle \chi\downarrow_N, \psi \rangle \neq 0 \}$ = "SET OF $\mathrm{Irr}(G)$ ABOVE ψ"

CLIFFORD THEOREM PICTURE

$$\{\chi_1, \ldots, \chi_i\} = \mathrm{Irr}(G \mid \psi) = \{\chi \in \mathrm{Irr}(G) \mid \langle \chi\downarrow_N, \psi \rangle \neq 0\}$$

IND $\uparrow$

$$\{\theta_1, \ldots, \theta_i\} = \mathrm{Irr}(G(\psi) \mid \psi)$$

$\uparrow$

$$\psi \in \mathrm{Irr}(N)$$

CLIFFORD CORRESPONDENCE:

$$\mathrm{Irr}(G(\psi) \mid \psi) \xrightarrow{\sim} \mathrm{Irr}(G \mid \psi), \quad H = G(\psi)$$

$$\theta \longmapsto \mathrm{IND}_H^G(\theta) = \frac{1}{|H|} \sum_{\substack{t \in G, \\ t^{-1}st \in H}} \theta(t^{-1}st)$$

(4) $\mathrm{Irr}(G_r)$ WHEN r IS EVEN

LET $r = 2\ell$, WE PICK:

- $G = G_r = GL_m(\mathcal{O}_r)$
- $N = K^{\ell}$

NEXT SUBSECTION:

(1) CONSTRUCT ELEMENTS OF $\mathrm{Irr}(N)$

(2) CONSTRUCT ELEMENTS OF $\mathrm{Irr}(H|\psi)$

(3) APPLY INDUCTION TO OBTAIN $\mathrm{Irr}(G|\psi)$

4.1 CONSTRUCT ELEMENTS OF $\mathrm{Irr}(N)$

LET F BE THE FRACTION FIELD OF $\mathcal{O}$, WE FIX AN ADDITIVE CHARACTER

$$\psi : F \to \mathbb{C}^\times \quad \text{WITH} \quad \mathrm{Ker}(\psi) = \mathcal{O}.$$

FOR EACH $M \in M_n(\mathcal{O}_r)$

DEFINE:
$$\psi_M : K^\ell \to \mathbb{C}^\times$$
$$I + \pi^\ell x \mapsto \psi(\pi^{-\ell}\, \mathrm{Tr}(Mx))$$

FACTS: ① $M_n(\mathcal{O}_\ell) \cong \mathrm{Irr}(K^\ell)$

$$M \mapsto \psi_M$$

(2) THE STABILIZER OF ψ_M IN G_r IS

$$G_r(\psi_M) = P_\ell^{-1}(C_{G_\ell}(M))$$

FOR $g \in G_r$, WE HAVE $\psi_M^{g^{-1}}(h) = \psi_M(g^{-1}hg) = \psi_{gMg^{-1}}(h)$

FOR ANY $h \in K^\ell$

BACK TO HISTORY / TERMINOLOGY

IRREDUCIBLE REGULAR REPRESENTATIONS

$$\{\chi \in \mathrm{Irr}(G_r \mid \psi_M) \text{ SUCH THAT } M \text{ IS A REGULAR MATRIX}\}$$

$M \bmod \pi$ IS REGULAR $\iff C_{\mathfrak{g}_1}(\bar{M})$ IS ABELIAN , EX: $\begin{bmatrix} 1 & 1 \\ 0 & 1+\pi \end{bmatrix}$ (HILL)

($M \bmod \pi = \bar{M}$; $\begin{bmatrix} 1 & 1 \\ 0 & 1+\pi \end{bmatrix} = M$)

STRONGLY SEMISIMPLE REPRESENTATIONS (HILL, 1995) $\Rightarrow$ THE SET OF $\mathrm{Irr}(G_r \mid \psi_M)$ WHERE M IS STRONGLY SEMISIMPLE MATRIX

SO $$M = S + m$$

WITH S SEMISIMPLE AND m NILPOTENT ELEMENT OF $Z(C_{M_m(\mathcal{O}_\ell)}(S))$

M IS A STABLE MATRIX IF M IS CONJUGATE TO A MATRIX

$A + \pi B$ WITH A IN JORDAN CANONICAL FORM

WITH B IN $Z(C_{M_m(\theta_\ell)}(A))$.

STABLE REPRESENTATIONS ARE ELEMENTS OF $\mathrm{Irr}(G_\ell \mid \psi_M)$

WITH M STABLE MATRIX.

FOR THE REST OF THE TALK, M IS A STABLE MATRIX

4.3: CONSTRUCT ELEMENTS OF $\mathrm{Irr}(H|V)$

THM (GALLAGHER CORRESPONDENCE)

IF ψ HAS AN EXTENSION $\hat{\psi}$ TO H (i.e. $\hat{\psi}\downarrow_N = \psi$) THEN

$$\mathrm{Irr}(H|\psi) = \{\hat{\psi}\cdot\sigma \mid \sigma \in \mathrm{Irr}(H/N)\}.$$

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THEOREM (M, 2023) LET $r = 2\ell$ AND χ BE A STABLE IRREDUCIBLE CHARACTER OF $G_r = GL_m(\mathcal{O}_r)$ ABOVE $\psi_M \in \mathrm{Irr}(K^\ell)$ THEN:

① ψ_M EXTENDS TO A CHARACTER $\tilde{\psi}_M$ OF THE STABILIZER $G_r(\psi_M)$.

② THERE IS AN IRREDUCIBLE CHARACTER σ OF $G_r(\psi_M)/K^\ell$ SUCH THAT

$$\chi = \mathrm{Ind}_{G_r(\psi_M)}^{G_r}(\tilde{\psi}_M \cdot \sigma)$$

(5) ODD CASE FOR $r = 2\ell - 1$, $\ell' = \ell - 1 = r - \ell$

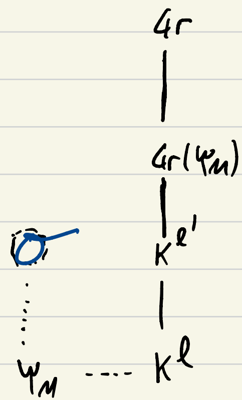
WE HAVE THE FOLLOWING CHAIN OF NORMAL SUBGROUPS OF G_r:

$$G_r \supset K^{\ell'} \supset K^{\ell} = I + \pi^{\ell} \mathfrak{g}_r$$

$K^{\ell'}$: NOT ABELIAN

K^{ℓ}: ABELIAN

PICTURE DIAGRAM



$G_r(\psi_M) = \rho_{\ell'}^{-1}(C_{G_r}(M))$

$\ell \neq \ell'$

* NOTE PICTURE IS DIFFERENT WHICH MAKES THINGS MORE COMPLICATED.

THEOREM (M): ASSUME THAT $\mathcal{O}$ HAS RESIDUE FIELD OF CHARACTERISTIC $p > 2$ AND $r = 2\ell - 1$. LET χ BE A SUPER STABLE IRREDUCIBLE CHARACTER OF G_r, $\chi \in \mathrm{Irr}(G_r \mid \sigma)$ AND $\sigma \in \mathrm{Irr}(K^{\ell'})$ THEN:

1. σ EXTENDS TO A CHARACTER $\tilde{\sigma}$ OF THE STABILIZER OF $G_r(\sigma)$.

2. THERE IS AN IRREDUCIBLE CHARACTER W OF $G_r(\sigma)/K^{\ell'}$ SUCH THAT

$$\chi = \mathrm{Ind}_{G_r(\sigma)}^{G_r} (\tilde{\sigma} W).$$

6 OPEN PROBLEMS AND OTHER RESULTS

- BEYOND THOSE CONSTRUCTIONS: IS THERE A UNIFORM CONSTRUCTION WHICH INCLUDES REGULAR AND THE STABLE REPRESENTATIONS? OR OTHER CLASSES?!

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NOTE $\mathbb{C}G \cong \bigoplus_{\chi \in Irr(G)} M_{\chi(1)}(\mathbb{C})$

LET $G_r = GL_n(\mathcal{O}_r)$ AND $G'_r = GL_n(\mathcal{O}'_r)$. WITH $\mathcal{O}_1 \cong \mathcal{O}'_1$

~ ONN'S CONJECTURE: $\mathbb{C}GL_n(\mathcal{O}_r) \cong \mathbb{C}GL_n(\mathcal{O}'_r)$. (2008)

- BEYOND COMPLEX REPRESENTATION:

IS $R G_r \cong R G'_r$ FOR A RING R?

WHEN $r=2$ WE KNOW:

YES: ① $R = K$ AN ALGEBRAIC CLOSED FIELD OF CHARACTERISTIC NOT p

$$K\,GL_m(\mathcal{O}_2) \simeq K\,GL_m(\mathcal{O}_2') \quad (M, 2021)$$

NO: ② $R = \mathbb{Z}$ AND $p \geq 5$ $\quad \mathbb{Z}\,GL_m(\mathcal{O}_2) \not\simeq \mathbb{Z}\,GL_m(\mathcal{O}_2')$ (M)

③ $R = F$ IS A FIELD OF CHAR. p AND $p \geq 2m$

P-MODULAR CASE $\quad F\,GL_m(\mathcal{O}_2) \not\simeq F\,GL_m(\mathcal{O}_2')$ (M)

WEAKER EQUIVALENCE ARE NOT TRUE: STABLY EQUIVALENT OF MORITA TYPE

THANKS