Kazhdan-Lusztig polynomials for p-adic Kac-Moody groups

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Summary

1. Kazhdan-Lusztig polynomials for p-adic reductive groups.
2. Passage to Kac-Moody groups.
3. The combinatorics of W^+ and its implications.

Setup

Object	Notation	Example
Connected split reductive group	$\mathbb{G}$	GL_n
Borel subgroup	$\mathbb{B}$	Upper triangular matrices
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Non-archimedean local field, ring of integer	$(\mathcal{K}, \mathcal{O}_{\mathcal{K}})$	$(\mathbb{Q}_p, \mathbb{Z}_p)$
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Definition: Iwahori subgroup

Set $G := \mathbb{G}(\mathcal{K})$. The **Iwahori subgroup** of G is:

$$I := \{g \in \mathbb{G}(\mathcal{O}_\mathcal{K}) \mid g \bmod \varpi \in \mathbb{B}(\Bbbk)\}.$$

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Theorem (Borel 1976)

$$\left\{ \begin{array}{c} \text{Smooth irreducible complex} \\ \text{representations of } G \\ \text{admitting a non-zero} \\ I\text{-invariant vector} \end{array} \right\}$$

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- a Bruhat order $\leq$,
- a compatible length function ℓ,
- a (quasi-)generating finite set S of simple reflections (of length 1).

Relation with the Iwahori-Hecke algebra

Theorem (Iwahori-Matsumoto 1965)

Setting $q = |\mathbb{k}|$, the Iwahori-Hecke algebra $\mathcal{H}_{\mathbb{C}}(G, I)$ has basis $(T_w)_{w \in \tilde{W}^a}$ with the relations:

$$\begin{aligned} T_{vw} &= T_v T_w \quad \text{if } \ell(v) + \ell(w) = \ell(vw), \\ T_s^2 &= (q-1)T_s + qT_1 \quad \text{for } s \in S. \end{aligned}$$

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- The R-polynomials: $(R_{w,v})$ such that $\overline{T_w} = \sum_{v \in \tilde{W}^a} R_{w,v}(q) T_v$.

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- The P-polynomials: $(P_{w,v})$ such that $C_w := q^{\frac{-\ell(w)}{2}} \sum_{v \leq w} P_{w,v}(q) T_v$ satisfies $C_w = \overline{C_w}$.

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- $\mathbb{G}$ a Kac-Moody group (eg $\mathbb{G} : R \mapsto \mathsf{SL_n}(R[t, t^{-1}]) \rtimes R^\times$)

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Difficulties

- No convolution product on $\mathcal{C}(I \backslash G / I, \mathbb{C})$,
- $Y \rtimes W^v$ has no Coxeter group structure (no finite set of simple reflections).

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- $Y^+ := \{y \in Y \mid \alpha(y) < 0 \text{ for finitely many } \alpha \in \Phi_+\}$,
- $Y^+ = Y$ if and only if $\mathbb{G}$ is reductive,
- Y^+ is W^v-stable, we define $W^+ := Y^+ \rtimes W^v$.

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Theorem

G admits a sub-semi-group $G^+ \supset I$ and a corresponding Iwahori-Hecke algebra $\mathcal{H}_{\mathbb{C}}(G^+, I)$ with a basis $(T_w)_{w \in W^+}$ indexed by W^+.

- Braverman, Kazhdan, Patnaik (Affine ADE, 2016),
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Issues

- W^+ is not a Coxeter group,
- $\mathcal{H}_{\mathbb{C}}(G^+, I)$ has no simple presentation,
- the Kazhdan-Lusztig involution is not defined.

The combinatorics of W^+

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Theorem: Welch (Affine ADE 2019) Hébert, P. (General Kac-Moody 2024)

Any element of W^+ admits a finite number of covers. In particular, intervals are finite.

A construction of R-polynomials

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1. $R_{x,y} = \sum R_{x,\mathfrak{c}}$ where $\mathfrak{c}$ runs over chains $y = x_0 < x_1 < \cdots < x_n = x$.
2. Combinatorial model for $R_{x,\mathfrak{c}}$ through Bruhat-Tits buildings.

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Strategy in the Kac-Moody case

1. The set of chains from y to x is still finite.
2. Combinatorial model for $R_{x,\mathfrak{c}}$ through masures (in progress).
3. Check that $R_{x,y} := \sum R_{x,\mathfrak{c}}$ are coefficients for an involution of $\mathcal{H}_{\mathbb{C}}(G^+, I)$ (difficult step).

Thank you for your attention !