The classification of the prime graphs of finite solvable cut/rational groups

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June 2024

Grant PID2020-113206GB-I00 funded by

MINISTERIO
DE CIENCIA, INNOVACIÓN
Y UNIVERSIDADES

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Graphs defined on groups

$$\{Groups\} \longrightarrow \{Graphs\}$$
$$G \longmapsto \Gamma(G)$$

Which properties of the group G can be recovered from the graph-theoretical properties of $\Gamma(G)$?

Examples:

- Commuting graph
- Cyclic graph
- Solvable conjugacy class graph
- Prime graph

Prime graph of a finite group

G a finite group

We construct the graph $\Gamma_{GK}(G) = \{V_{vertices}, E_{edges}\}$ as follows:

- $V = \pi(G) = \{\text{primes dividing } |G|\}$
- $p, q \in V$ different,

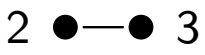
$$p - q \in E \iff \exists\, g \in G : \ |g| = pq$$

Definition (Prime graph or GK-graph)

$\Gamma_{GK}(G)$ is the *prime graph* or *Gruenberg-Kegel graph* of G.

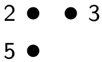
Example:

C_6



S_3

Example: A_5 has prime graph



Some results about prime graphs:

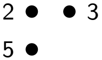
- Gruenberg-Kegel: G finite solvable group,

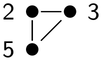
  $\Gamma_{GK}(G)$ disconected $\iff$ G Frobenius or 2-Frobenius.
- Lucido: G finite group such that $\Gamma_{GK}(G)$ is a tree $\implies |\pi(G)| \leq 8$.
- Gavrilyuk-Khramtsov-Kondrat'ev-Maslova: Characterization of the prime graphs of finite groups with at most 5 vertices.
- Gruber-Keller-Lewis-Naughton-Strasser: Characterization of the prime graphs of finite solvable groups.

Characterization of the prime graphs of finite solvable groups

Theorem (Gruber-Keller-Lewis-Naughton-Strasser)

A graph Γ is isomorphic to the prime graph of a finite solvable group if and only if its complement $\overline{\Gamma}$ is 3-colorable and triangle-free.

Example: $\Gamma =$ 

$\Longrightarrow \overline{\Gamma} =$ 

, $\Gamma \neq \Gamma_{GK}(G)$ for G finite solvable

- Question 1: What are the prime graphs of finite solvable groups satisfying some property P?

- Question 2 (Realizability): Given a graph Γ, is there a finite solvable group G satisfying some property P such that $\Gamma = \Gamma_{GK}(G)$?

$P =$ rational/cut.

Rational/Cut = Inverse-semirational

G a finite group

$g \in G$ *rational* $\overset{def}{\Longleftrightarrow}$ $(\forall h$ generator of $\langle g \rangle, h \sim g)$

Definition (Rational)

A group G is *rational* if every element of G is rational.

Example: S_n is rational for every $n \in \mathbb{N}$.

$g \in G$ *inverse semi-rational* $\overset{def}{\Longleftrightarrow}$ $(\forall h$ generator of $\langle g \rangle, h \sim g$ or $h \sim g^{-1})$

Definition (Cut = Inverse-semirational)

A group G is *cut* if every element of G is inverse-semirational.

Example: The monster group M is cut.

$$\{\text{Rational groups}\} << \{\text{Cut groups}\}$$

Remark (Bächle-Caicedo-Jespers-Maheshwary)

Among groups of order at most 1023:

- *78.55% are cut groups*
- *0.52% are rational.*

$g \in G$,

$$B_G(g) = \frac{N_G(\langle g \rangle)}{C_G(g)} \lesssim Aut(\langle g \rangle)$$

$\varphi =$ Euler's totient function

- g rational $\iff |B_G(g)| = \varphi(|g|)$
- g inverse-semirational $\iff \begin{cases} g \text{ rational or} \\ |B_G(g)| = \varphi(|g|)/2 \text{ and } g \not\sim g^{-1} \end{cases}$

The classification of the prime graphs of finite solvable rational/cut groups

Example: There are no finite cut groups having $(3-7)$ as prime graph.

G cut and $g \in G$ of order 21 $\implies 2 \mid |B_G(g)|$

- Is it possible to classify the prime graphs of finite rational/cut groups?

  Difficult to approach! Each prime p divides the order of S_p.

- What about finite solvable rational/cut groups?

Theorem (Gow)

Let G be a finite solvable rational group. Then $\pi(G) \subseteq \{2,3,5\}$.

Theorem (Bächle)

Let G be a finite solvable cut group. Then $\pi(G) \subseteq \{2,3,5,7\}$.

The classification of Bächle-Kiefer-Maheshwary-del Rio

GK-graphs of solvable finite cut/rational groups:

- 1 vertex:

a)


C_2

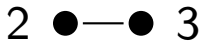
b)


C_3

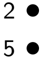
- 2 vertices:

c)

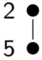

S_3

d)


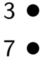
$S_3 \times C_2$

e)


$K = C_5^2 \rtimes_{Fr} Q_8$

f)


$K \times C_2$

g)


$C_7 \rtimes_{Fr} C_3$

The classification of Bächle-Kiefer-Maheshwary-del Rio

GK-graphs of solvable finite cut/rational groups:

- 1 vertex:

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- 2 vertices:

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S_3

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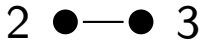
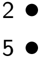
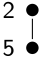


$S_3 \times C_2$

e)


$K = C_5^2 \rtimes_{Fr} Q_8$

f)


$K \times C_2$

g)

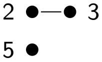

$C_7 \rtimes_{Fr} C_3$

The classification of Bächle-Kiefer-Maheshwary-del Rio

GK-graphs of solvable finite cut/rational groups:

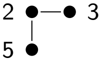
- 3 vertices:

h)



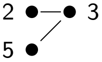
$L = C_5^2 \rtimes_{Fr} Dic_3$

i)



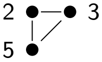
$L \times C_2$

j)



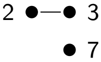
$K \times C_3$

k)



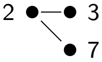
$K \times S_3$

l)



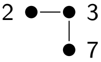
$M = C_7 \rtimes_{Fr} C_6$

m)



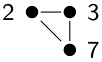
$M \times C_2$

n)



$M \times C_3$

o)



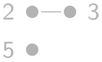
$M \times S_3$

The classification of Bächle-Kiefer-Maheshwary-del Rio

GK-graphs of solvable finite cut/rational groups:

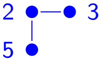
- 3 vertices:

h)



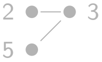
$L = C_5^2 \rtimes_{Fr} Dic_3$

i)



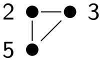
?

j)



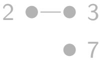
$K \times C_3$

k)



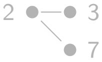
$K \times S_3$

l)



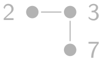
$M = C_7 \rtimes_{Fr} C_6$

m)



$M \times C_2$

n)



$M \times C_3$

o)

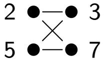
$M \times S_3$

The classification of Bächle-Kiefer-Maheshwary-del Rio

GK-graphs of solvable finite cut/rational groups:

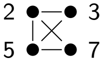
- 4 vertices:

p)



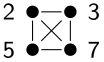
$N = K \times (C_7 \rtimes_{Fr} C_3)$

q)



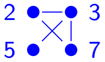
$N \times C_2$

r)



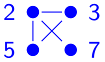
$N \times S_3$

s)



?

t)



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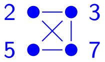
u)



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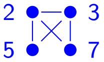
v)



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The classification of Bächle-Kiefer-Maheshwary-del Rio

GK-graphs of solvable finite cut/rational groups:

- 4 vertices:

p)

$N = K \times (C_7 \rtimes_{Fr} C_3)$

q)

$N \times C_2$

r)

$N \times S_3$

s)

?

t)

?

u)

?

v)

?

Summarizing...

To complete the classification we have to answer the following questions.

For rational groups:

Question (Bächle-Kiefer-Maheshwary-del Rio)

Is $(3 - 2 - 5)$ *the GK-graph of a finite solvable rational group?*

For cut groups:

Question (Bächle-Kiefer-Maheshwary-del Rio)

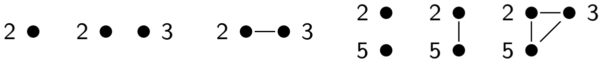
Which of the four graphs $s), t), u)$ *and* $v)$ *are realizable as the GK-graph of some finite solvable cut group?*

For rational groups:

Theorem (CD, Garcia-Lucas, del Río)

$(3 - 2 - 5)$ is not the GK-graph of a finite solvable rational group.

The GK-graphs of non-trivial finite solvable rational groups are precisely the following:



Corollary

If G is a finite solvable rational group of order divisible by 15, then G has elements of order 6, 10 and 15.

For cut groups:

At the moment...we do not know much. But we have the following conjecture.

Conjecture

A finite solvable cut group having GK-graph s), t), u) or v) has Fitting length at least 5*.*

This is true for *s*).

Equivalently,

Conjecture

The GK-graphs of finite solvable cut groups with Fitting length at most 4 are the graphs a) - r).

Thank you!

Bibliography

A. Bächle, *Integral group rings of solvable groups with trivial central units*, Forum Math. **30** (2018), no. 4, 845–855.

A. Bächle, M. Caicedo, E. Jespers, and S. Maheshwary, *Global and local properties of finite groups with only finitely many central units in their integral group ring*, 11 pages, submitted, `arXiv:1808.03546v2[math.RA]`.

A. Bächle, A. Kiefer, S. Maheshwary, and Á. del Río, *Gruenberg–kegel graphs: Cut groups, rational groups and the prime graph question*, Forum Mathematicum **35** (2023), no. 2, 409–429.

P.J. Cameron, *Graphs defined on groups*, International Journal of Group Theory **11** (2022), no. 2, 53–107, doi: 10.22108/ijgt.2021.127679.1681.

A. L. Gavrilyuk, I. V. Khramtsov, A. S. Kondrat'ev, and N. V. Maslova, *On realizability of a graph as the prime graph of a finite group*, Sib. Èlektron. Mat. Izv. **11** (2014), 246–257.

A. Gruber, T. Mi. Keller, M. L. Lewis, K. Naughton, and B. Strasser, *A characterization of the prime graphs of solvable groups*, J. Algebra **442** (2015), 397–422.

R. Gow, *Groups whose characters are rational-valued*, J. Algebra **40** (1976), no. 1, 280–299.

P. Hegedűs, *Structure of solvable rational groups*, Proc. London Math. Soc. (3) **90** (2005), no. 2, 439–471.

M. S Lucido, *Groups in which the prime graph is a tree*, Boll. Unione Mat. Ital. Sez. B Artic. Ric. Mat. (8) **5** (2002), no. 1, 131–148, `http://www.bdim.eu/item?id=BUMI_2002_8_5B_1_131_0&fmt=pdf`.

N. V. Maslova, *On the gruenberg – kegel graphs of finite groups*, 2016.

J. S. Williams, *Prime graph components of finite groups*, J. Algebra **69** (1981), no. 2, 487–513.