Chiralisation of reduction by stages

Thibault Juillard (project with Naoki Genra)

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Université Paris-Saclay, Institut de Mathématiques d'Orsay

Plan

Introduction

Reduction by stages for Slodowy slices

Reduction by stages for finite W-algebra

Quantisation and chiralisation of Kraft and Procesi's Theorems

Introduction

Quantisation and chiralisation

We work over $\mathbf{C}$.

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Associative algebras

$$\begin{array}{ccc} \mathscr{U}(\mathfrak{g}) & \xrightarrow{\text{gr}} & \mathbf{C}[\mathfrak{g}^*] \\ \big\downarrow {\scriptstyle \text{Hamiltonian reduction}} & & \big\downarrow {\scriptstyle \text{Hamiltonian reduction}} \\ \mathscr{U}(\mathfrak{g}, f) & \xrightarrow[\text{gr}]{} & \mathbf{C}[S_f] \end{array}$$

Poisson algebras

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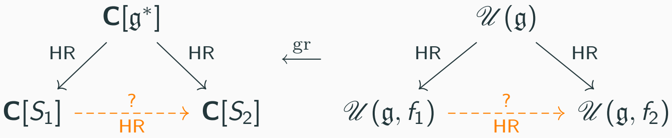

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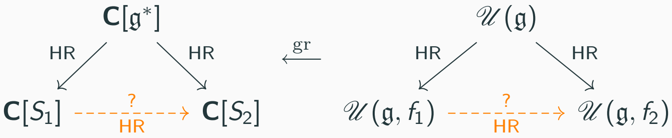

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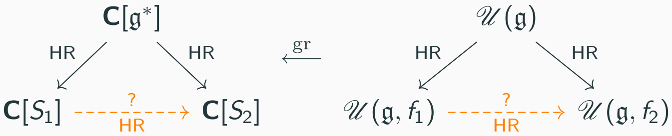

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Positive answers to these questions (Genra-J. arXiv:2212.06022).

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Vertex algebras

Kac-Moody algebra

$\mathscr{V}^{\kappa}(\mathfrak{g}) \xrightarrow{\mathrm{gr}} \mathbf{C}[J_\infty \mathfrak{g}^*]$

Hamiltonian reduction ↓ Hamiltonian reduction ↓

$\mathscr{W}^{\kappa}(\mathfrak{g}, f) \xrightarrow{\mathrm{gr}} \mathbf{C}[J_\infty S_f]$

Affine W-algebra

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Vertex algebras (left column); Poisson vertex algebras (right column)

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$\downarrow$ Hamiltonian reduction $\quad\quad$ $\downarrow$ Hamiltonian reduction

$\mathscr{W}^{\kappa}(\mathfrak{g}, f)$ Affine W-algebra $\xrightarrow{\text{gr}}$ $\mathbf{C}[J_\infty S_f]$

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In progress (Genra-J.)

Reduction by stages for Slodowy slices

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Its Poisson structure is induced by Hamiltonian reduction.

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Remark

Explicit description would require to compute the ring of invariants $\left(\mathbf{C}[X]/I\right)^M$: difficult.

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Then, M is the unipotent radical in a Borel subgroup of G.

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$\mathfrak{g} = \mathfrak{sl}_n$ and $f = E_{2,1} + E_{3,2} + \cdots + E_{n,n-1}$.

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The conditions imply $G \cdot f_1 \subseteq \overline{G \cdot f_2}$.

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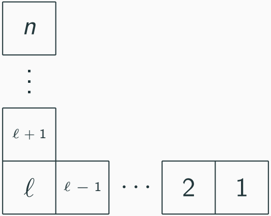

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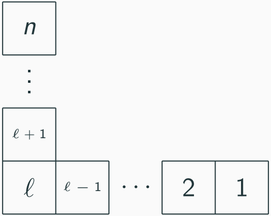

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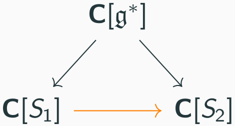

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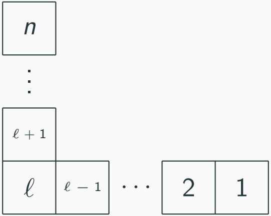

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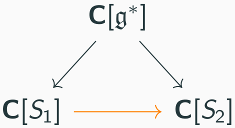

Other examples in classical and exceptional types.

Reduction by stages for finite W-algebra

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Quantisation and chiralisation of Kraft and Procesi's Theorems

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Remark

Reinterpretation of the row elimination rule of Kraft and Procesi.

Rule used to classify minimal degenerations mentioned in Gwyn's talk.

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- $\mathfrak{g} \coloneqq \mathfrak{gl}_n$.
- f_2 associated to partition $(\lambda_1, \dots, \lambda_k, \underline{\lambda}_0)$.
- $\underline{\lambda}_0$ partition of $n_0 < n$: $f_0 \in \mathfrak{gl}_{n_0}$ associated to partition $\underline{\lambda}_0$.
- f_1 associated to partition $(\lambda_1, \dots, \lambda_k, 1^{n_0})$.

Theorem (J.)

1. $\mathscr{U}(\mathfrak{gl}_n, f_1)$ is the Hamiltonian reduction of $\mathscr{U}(\mathfrak{gl}_n, f_2)$.
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Conjecture ★

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Conjecture ★

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Quantisation and chiralisation

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Theorem (J.)

1. $\mathscr{U}(\mathfrak{gl}_n, f_1)$ is the Hamiltonian reduction of $\mathscr{U}(\mathfrak{gl}_n, f_2)$.
2. There is an embedding $\mathscr{U}(\mathfrak{gl}_{n_0}, f_0) \hookrightarrow \mathscr{U}(\mathfrak{gl}_n, f_2)$.

Conjecture ★

1. $\mathscr{W}^\kappa(\mathfrak{gl}_n, f_1)$ is the Hamiltonian reduction of $\mathscr{W}^\kappa(\mathfrak{gl}_n, f_2)$.
2. There is an embedding $\mathscr{W}^{\kappa'}(\mathfrak{gl}_{n_0}, f_0) \hookrightarrow \mathscr{W}^\kappa(\mathfrak{gl}_n, f_2)$.

Thank you for your attention!