The Proce Γ-fixed point punctual Hilbert scheme in $\mathbb{C}^2$

Raphaël Paegelow (joint work with Gwyn Bellamy)

IMAG, University of Montpellier

June 6, 2024

I - Main players

I - Main players

Take $n \geq 1$.

I - Main players

Take $n \geq 1$.

- $\mathcal{H}_n := \mathrm{Hilb}_n(\mathbb{C}^2) = \{I \subset \mathbb{C}[x,y] \big| I$ is an ideal and $\dim(\mathbb{C}[x,y]/I) = n\}$

I - Main players

Take $n \geq 1$.

- $\mathcal{H}_n := \mathrm{Hilb}_n(\mathbb{C}^2) = \{I \subset \mathbb{C}[x,y] \big| I$ is an ideal and $\dim(\mathbb{C}[x,y]/I) = n\}$

- [Fo68, Thm. 2.9] : $\mathcal{H}_n$ is a smooth algebraic variety of dimension $2n$.

I - Main players

Take $n \geq 1$.

- $\mathcal{H}_n := \mathrm{Hilb}_n(\mathbb{C}^2) = \{I \subset \mathbb{C}[x,y] \big| I \text{ is an ideal and } \dim(\mathbb{C}[x,y]/I) = n\}$

- [Fo68, Thm. 2.9] : $\mathcal{H}_n$ is a smooth algebraic variety of dimension $2n$.
- Hilbert-Chow morphism

$$\sigma_n : \begin{array}{ccc} \mathcal{H}_n & \to & (\mathbb{C}^2)^n/\mathfrak{S}_n \\ I & \mapsto & \sum_{p \in V(I)} \dim\left((\mathbb{C}[x,y]/I)_p\right)[p] \end{array}$$ [(1)]

(1). $[p]$ denotes the class of p in $(\mathbb{C}^2)^n/\mathfrak{S}_n$

I - Main players

- $\mathcal{X}_n :=$ reduced scheme associated with $\mathcal{H}_n \times_{(\mathbb{C}^2)^n/\mathfrak{S}_n} (\mathbb{C}^2)^n$

$$\begin{array}{ccc} \mathcal{H}_n \times_{(\mathbb{C}^2)^n/\mathfrak{S}_n} (\mathbb{C}^2)^n & \xrightarrow{f_n} & (\mathbb{C}^2)^n \\ \downarrow{\rho_n} & \lrcorner & \downarrow{\pi_n} \\ \mathcal{H}_n & \xrightarrow[\sigma_n]{} & (\mathbb{C}^2)^n/\mathfrak{S}_n \end{array}$$

I - Main players

- $\mathcal{X}_n :=$ reduced scheme associated with $\mathcal{H}_n \times_{(\mathbb{C}^2)^n/\mathfrak{S}_n} (\mathbb{C}^2)^n$

$$
\begin{array}{ccc}
\mathcal{H}_n \times_{(\mathbb{C}^2)^n/\mathfrak{S}_n} (\mathbb{C}^2)^n & \xrightarrow{\;f_n\;} & (\mathbb{C}^2)^n \\
\Big\downarrow{\scriptstyle \rho_n} \quad \lrcorner & & \Big\downarrow{\scriptstyle \pi_n} \\
\mathcal{H}_n & \xrightarrow[\;\sigma_n\;]{} & (\mathbb{C}^2)^n/\mathfrak{S}_n
\end{array}
$$

  - $\mathcal{X}_n$ is an algebraic variety.

I - Main players

- $\mathcal{X}_n$:= reduced scheme associated with $\mathcal{H}_n \times_{(\mathbb{C}^2)^n/\mathfrak{S}_n} (\mathbb{C}^2)^n$

$$\begin{array}{ccc} \mathcal{H}_n \times_{(\mathbb{C}^2)^n/\mathfrak{S}_n} (\mathbb{C}^2)^n & \xrightarrow{f_n} & (\mathbb{C}^2)^n \\ \downarrow \rho_n & \lrcorner & \downarrow \pi_n \\ \mathcal{H}_n & \xrightarrow[\sigma_n]{} & (\mathbb{C}^2)^n/\mathfrak{S}_n \end{array}$$

  - ▶ $\mathcal{X}_n$ is an algebraic variety.
  - ▶ [H03, Thm. $5.2.1$] The morphism $\rho_n : \mathcal{X}_n \to \mathcal{H}_n$ is flat (and by construction finite of rank $n!$).

I - Main players

- $\mathcal{X}_n :=$ reduced scheme associated with $\mathcal{H}_n \times_{(\mathbb{C}^2)^n/\mathfrak{S}_n} (\mathbb{C}^2)^n$

$$\begin{array}{ccc} \mathcal{H}_n \times_{(\mathbb{C}^2)^n/\mathfrak{S}_n} (\mathbb{C}^2)^n & \xrightarrow{f_n} & (\mathbb{C}^2)^n \\ \downarrow \rho_n & \lrcorner & \downarrow \pi_n \\ \mathcal{H}_n & \xrightarrow[\sigma_n]{} & (\mathbb{C}^2)^n/\mathfrak{S}_n \end{array}$$

  - $\mathcal{X}_n$ is an algebraic variety.
  - [H03, Thm. $5.2.1$] The morphism $\rho_n : \mathcal{X}_n \to \mathcal{H}_n$ is flat (and by construction finite of rank $n!$).

- $\mathscr{P}^n := \rho_{n*}\mathcal{O}_{\mathcal{X}_n}$ is a locally free sheaf over $\mathcal{H}_n$ of rank $n! \rightsquigarrow$ the Procesi bundle.

I - Main players

Let $\mathrm{GL}_2(\mathbb{C})$ act naturally on $\mathbb{C}^2$

$\rightsquigarrow$ $\mathrm{GL}_2(\mathbb{C}) \curvearrowright \mathcal{H}_n$

$\rightsquigarrow$ $\mathrm{GL}_2(\mathbb{C}) \curvearrowright (\mathbb{C}^2)^n$

I - Main players

$$\mathrm{GL}_2(\mathbb{C}) \curvearrowright \mathcal{H}_n$$

Let $\mathrm{GL}_2(\mathbb{C})$ act naturally on $\mathbb{C}^2$ $\rightsquigarrow$ and $\mathfrak{S}_n$ act trivially on $\mathcal{H}_n$.

$$\mathrm{GL}_2(\mathbb{C}) \curvearrowright (\mathbb{C}^2)^n$$

I - Main players

Let $\mathrm{GL}_2(\mathbb{C})$ act naturally on $\mathbb{C}^2$ $\rightsquigarrow$ $\mathrm{GL}_2(\mathbb{C}) \curvearrowright \mathcal{H}_n$ and $\mathrm{GL}_2(\mathbb{C}) \curvearrowright (\mathbb{C}^2)^n$ and $\mathfrak{S}_n$ act trivially on $\mathcal{H}_n$.

The morphisms ρ_n, σ_n, π_n and f_n are all $(\mathfrak{S}_n \times \mathrm{GL}_2(\mathbb{C}))$-equivariant.

$$\begin{array}{ccc} \mathcal{H}_n \times_{(\mathbb{C}^2)^n/\mathfrak{S}_n} (\mathbb{C}^2)^n & \xrightarrow{f_n} & (\mathbb{C}^2)^n \\ \downarrow \rho_n & \lrcorner & \downarrow \pi_n \\ \mathcal{H}_n & \xrightarrow[\sigma_n]{} & (\mathbb{C}^2)^n/\mathfrak{S}_n \end{array}$$

I - Main players

- Let Γ be a finite subgroup of $\mathrm{SL}_2(\mathbb{C})$ of order ℓ (ADE).

I - Main players

- Let Γ be a finite subgroup of $\mathrm{SL}_2(\mathbb{C})$ of order ℓ (ADE).

  If $\mathcal{H}_n^{\Gamma,\chi} := \{I \in \mathcal{H}_n^{\Gamma} | \mathrm{Tr}\,(\mathbb{C}[x,y]/I) = \chi\}$

I - Main players

- Let Γ be a finite subgroup of $\mathrm{SL}_2(\mathbb{C})$ of order ℓ (ADE).

  If $\mathcal{H}_n^{\Gamma,\chi} := \{I \in \mathcal{H}_n^{\Gamma} | \mathrm{Tr}\,(\mathbb{C}[x,y]/I) = \chi\}$, and

  if $\mathcal{A}_\Gamma^n :=$ { all characters χ of Γ of degree n such that $\mathcal{H}_n^{\Gamma,\chi} \neq \emptyset\}$,

- Let Γ be a finite subgroup of $\mathrm{SL}_2(\mathbb{C})$ of order ℓ (ADE).

  If $\mathcal{H}_n^{\Gamma,\chi} := \{I \in \mathcal{H}_n^{\Gamma} | \mathrm{Tr}\,(\mathbb{C}[x,y]/I) = \chi\}$, and

  if $\mathcal{A}_\Gamma^n :=$ { all characters χ of Γ of degree n such that $\mathcal{H}_n^{\Gamma,\chi} \neq \emptyset\}$, then

$$\boxed{\mathcal{H}_n^{\Gamma} = \coprod_{\chi \in \mathcal{A}_\Gamma^n} \mathcal{H}_n^{\Gamma,\chi}}$$

I - Main players

Take $\chi \in \mathcal{A}_\Gamma^n$.

I - Main players

Take $\chi \in \mathcal{A}_\Gamma^n$.

$$\mathscr{P}^n_{|\mathcal{H}_n^{\Gamma,\chi}} \rightsquigarrow \text{a vector bundle over } \mathcal{H}_n^{\Gamma,\chi} \text{ whose fibers are } (\mathfrak{S}_n \times \Gamma)\text{-modules.}$$

I - Main players

Take $\chi \in \mathcal{A}_\Gamma^n$.

$\mathscr{P}^n_{|\mathcal{H}_n^{\Gamma,\chi}} \rightsquigarrow$ a vector bundle over $\mathcal{H}_n^{\Gamma,\chi}$ whose fibers are $(\mathfrak{S}_n \times \Gamma)$-modules.

Our Goal :

Study the fiber $\mathscr{P}^n_{|I}$ as a $(\mathfrak{S}_n \times \Gamma)$-module, $\forall I \in \mathcal{H}_n^{\Gamma,\chi}$

II - Reduction theorem

II - Reduction theorem

Fix $\chi \in \mathcal{A}_{\Gamma}^{n}$.

II - Reduction theorem

Fix $\chi \in \mathcal{A}_\Gamma^n$.

$$r_\chi := \frac{1}{2}\dim\left(\mathcal{H}_n^{\Gamma,\chi}\right)$$

(2). $\mathrm{reg}(\Gamma)$ denotes the character of the regular representation of Γ

II - Reduction theorem

Fix $\chi \in \mathcal{A}_\Gamma^n$.

$r_\chi := \frac{1}{2}\dim\left(\mathcal{H}_n^{\Gamma,\chi}\right)$ and $\chi_0 := \chi - r_\chi \mathrm{reg}(\Gamma)$ [(2)]

(2). $\mathrm{reg}(\Gamma)$ denotes the character of the regular representation of Γ

II - Reduction theorem

Fix $\chi \in \mathcal{A}_\Gamma^n$.

$r_\chi := \frac{1}{2}\dim\left(\mathcal{H}_n^{\Gamma,\chi}\right)$ and $\chi_0 := \chi - r_\chi \mathrm{reg}(\Gamma)$ [2]

$g_\Gamma :=$ degree of χ_0

(2). $\mathrm{reg}(\Gamma)$ denotes the character of the regular representation of Γ

II - Reduction theorem

Fix $\chi \in \mathcal{A}_\Gamma^n$.

$r_\chi := \frac{1}{2}\dim\left(\mathcal{H}_n^{\Gamma,\chi}\right)$ and $\chi_0 := \chi - r_\chi \mathrm{reg}(\Gamma)$ [2]

$g_\Gamma :=$ degree of χ_0

The I.C. $\mathcal{H}_{g_\Gamma}^{\Gamma,\chi_0}$ is zero-dimensional

(2). $\mathrm{reg}(\Gamma)$ denotes the character of the regular representation of Γ

Fix $\chi \in \mathcal{A}_\Gamma^n$.

$r_\chi := \frac{1}{2}\dim\left(\mathcal{H}_n^{\Gamma,\chi}\right)$ and $\chi_0 := \chi - r_\chi \mathrm{reg}(\Gamma)$ (2)

$g_\Gamma :=$ degree of χ_0

The I.C. $\mathcal{H}_{g_\Gamma}^{\Gamma,\chi_0}$ is zero-dimensional $\rightsquigarrow \mathcal{H}_{g_\Gamma}^{\Gamma,\chi_0} = \{I_{\chi_0}\}$.

(2). $\mathrm{reg}(\Gamma)$ denotes the character of the regular representation of Γ

II - Reduction theorem

Fix $\chi \in \mathcal{A}_\Gamma^n$.

$r_\chi := \frac{1}{2}\dim\left(\mathcal{H}_n^{\Gamma,\chi}\right)$ and $\chi_0 := \chi - r_\chi \mathrm{reg}(\Gamma)$ (2)

$g_\Gamma :=$ degree of χ_0

The I.C. $\mathcal{H}_{g_\Gamma}^{\Gamma,\chi_0}$ is zero-dimensional $\rightsquigarrow \mathcal{H}_{g_\Gamma}^{\Gamma,\chi_0} = \{I_{\chi_0}\}$.

Take $(p_1, \ldots, p_{r_\chi}) \in (\mathbb{C}^2 \setminus \{(0,0)\})^{r_\chi}$ such that

$$\forall (i,j) \in [\![1, r_\chi]\!]^2, i \neq j \Rightarrow \Gamma p_i \cap \Gamma p_j = \emptyset$$

(2). $\mathrm{reg}(\Gamma)$ denotes the character of the regular representation of Γ

II - Reduction theorem

Fix $\chi \in \mathcal{A}_\Gamma^n$.

$r_\chi := \frac{1}{2}\dim\left(\mathcal{H}_n^{\Gamma,\chi}\right)$ and $\chi_0 := \chi - r_\chi \mathrm{reg}(\Gamma)$ (2)

$g_\Gamma :=$ degree of χ_0

The I.C. $\mathcal{H}_{g_\Gamma}^{\Gamma,\chi_0}$ is zero-dimensional $\rightsquigarrow \mathcal{H}_{g_\Gamma}^{\Gamma,\chi_0} = \{I_{\chi_0}\}$.

Take $(p_1, \ldots, p_{r_\chi}) \in (\mathbb{C}^2 \setminus \{(0,0)\})^{r_\chi}$ such that

$$\forall (i,j) \in [\![1, r_\chi]\!]^2, i \neq j \Rightarrow \Gamma p_i \cap \Gamma p_j = \emptyset$$

$q := (\Gamma p_1, \ldots, \Gamma p_{r_\chi}) \in (\mathbb{C}^2)^{r_\chi \ell}$

(2). $\mathrm{reg}(\Gamma)$ denotes the character of the regular representation of Γ

II - Reduction theorem

Fix $\chi \in \mathcal{A}_\Gamma^n$.

$r_\chi := \frac{1}{2}\dim\left(\mathcal{H}_n^{\Gamma,\chi}\right)$ and $\chi_0 := \chi - r_\chi \mathrm{reg}(\Gamma)$ (2)

$g_\Gamma :=$ degree of χ_0

The I.C. $\mathcal{H}_{g_\Gamma}^{\Gamma,\chi_0}$ is zero-dimensional $\rightsquigarrow \mathcal{H}_{g_\Gamma}^{\Gamma,\chi_0} = \{I_{\chi_0}\}$.

Take $(p_1, \ldots, p_{r_\chi}) \in (\mathbb{C}^2 \setminus \{(0,0)\})^{r_\chi}$ such that

$$\forall (i,j) \in [\![1, r_\chi]\!]^2, i \neq j \Rightarrow \Gamma p_i \cap \Gamma p_j = \emptyset$$

$q := (\Gamma p_1, \ldots, \Gamma p_{r_\chi}) \in (\mathbb{C}^2)^{r_\chi \ell}$

$p := (0, q) \in (\mathbb{C}^2)^n$

(2). $\mathrm{reg}(\Gamma)$ denotes the character of the regular representation of Γ

II - Reduction theorem

Fix $\chi \in \mathcal{A}_\Gamma^n$.

$r_\chi := \frac{1}{2}\dim\left(\mathcal{H}_n^{\Gamma,\chi}\right)$ and $\chi_0 := \chi - r_\chi \mathrm{reg}(\Gamma)$ (2)

$g_\Gamma :=$ degree of χ_0

The I.C. $\mathcal{H}_{g_\Gamma}^{\Gamma,\chi_0}$ is zero-dimensional $\rightsquigarrow \mathcal{H}_{g_\Gamma}^{\Gamma,\chi_0} = \{I_{\chi_0}\}$.

Take $(p_1, \ldots, p_{r_\chi}) \in (\mathbb{C}^2 \setminus \{(0,0)\})^{r_\chi}$ such that

$$\forall (i,j) \in [\![1, r_\chi]\!]^2, i \neq j \Rightarrow \Gamma p_i \cap \Gamma p_j = \emptyset$$

$q := (\Gamma p_1, \ldots, \Gamma p_{r_\chi}) \in (\mathbb{C}^2)^{r_\chi \ell}$

$p := (0, q) \in (\mathbb{C}^2)^n$

$S_p :=$ stabilizer of p in $\mathfrak{S}_n \times \Gamma$.

(2). $\mathrm{reg}(\Gamma)$ denotes the character of the regular representation of Γ

II - Reduction theorem

Theorem

There exists an isomorphism $\text{₮} : S_p \xrightarrow{\sim} \mathfrak{S}_{g_\Gamma} \times \Gamma$ *which endows* $\mathscr{P}^{\mathrm{g}_\Gamma}_{|I_{\chi 0}}$ *with a* S_p*-module structure, such that for each* $I \in \mathcal{H}_n^{\Gamma,\chi}$*,*

$$\left[\mathscr{P}^{n}_{|I}\right]_{\mathfrak{S}_n\times\Gamma} = \left[\mathrm{Ind}_{S_p}^{\mathfrak{S}_n\times\Gamma}\left(\mathscr{P}^{\mathrm{g}_\Gamma}_{|I_{\chi 0}}\right)\right]_{\mathfrak{S}_n\times\Gamma}.$$

III - Type A & Combinatorics

III - Type A & Combinatorics

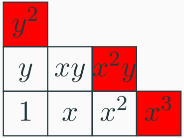
$\mathbb{T}_1 :=$ maximal diagonal torus of $\mathrm{SL}_2(\mathbb{C})$.

III - Type A & Combinatorics

$\mathbb{T}_1 :=$ maximal diagonal torus of $\mathrm{SL}_2(\mathbb{C})$.

$\mathcal{H}_n^{\mathbb{T}_1}$ are parametrized by partitions of size n.

For example if $\lambda = (3,2) \vdash 5$



Then $I_{(3,2)} = \langle x^3, x^2y, y^2 \rangle \in \mathcal{H}_5^{\mathbb{T}_1}$.

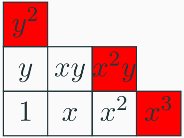
III - Type A & Combinatorics

$\mathbb{T}_1 :=$ maximal diagonal torus of $\mathrm{SL}_2(\mathbb{C})$.

$\mathcal{H}_n^{\mathbb{T}_1}$ are parametrized by partitions of size n.

Fix $\ell \geq 1$.

For example if $\lambda = (3,2) \vdash 5$



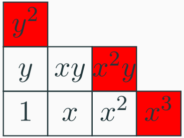
Then $I_{(3,2)} = \langle x^3, x^2y, y^2 \rangle \in \mathcal{H}_5^{\mathbb{T}_1}$.

III - Type A & Combinatorics

$\mathbb{T}_1 :=$ maximal diagonal torus of $\mathrm{SL}_2(\mathbb{C})$.

$\mathcal{H}_n^{\mathbb{T}_1}$ are parametrized by partitions of size n.

For example if $\lambda = (3,2) \vdash 5$



Then $I_{(3,2)} = \langle x^3, x^2y, y^2 \rangle \in \mathcal{H}_5^{\mathbb{T}_1}$.

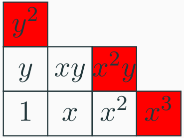
Fix $\ell \geq 1$.
ζ_ℓ: a primitive root of unity.

III - Type A & Combinatorics

$\mathbb{T}_1 :=$ maximal diagonal torus of $\mathrm{SL}_2(\mathbb{C})$.

$\mathcal{H}_n^{\mathbb{T}_1}$ are parametrized by partitions of size n.

For example if $\lambda = (3,2) \vdash 5$



Then $I_{(3,2)} = \langle x^3, x^2y, y^2 \rangle \in \mathcal{H}_5^{\mathbb{T}_1}$.

Fix $\ell \geq 1$.

ζ_ℓ: a primitive root of unity.

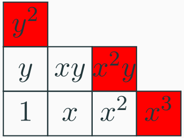
$\Gamma = \mu_\ell < \mathbb{T}_1$ generated by $\omega_\ell = \mathrm{diag}(\zeta_\ell, \zeta_\ell^{-1})$

III - Type A & Combinatorics

$\mathbb{T}_1 :=$ maximal diagonal torus of $\mathrm{SL}_2(\mathbb{C})$.

$\mathcal{H}_n^{\mathbb{T}_1}$ are parametrized by partitions of size n.

For example if $\lambda = (3,2) \vdash 5$



Then $I_{(3,2)} = \langle x^3, x^2y, y^2 \rangle \in \mathcal{H}_5^{\mathbb{T}_1}$.

Fix $\ell \geq 1$.

ζ_ℓ: a primitive root of unity.

$\Gamma = \mu_\ell < \mathbb{T}_1$ generated by $\omega_\ell = \mathrm{diag}(\zeta_\ell, \zeta_\ell^{-1})$ and $\tau_\ell : \begin{array}{ccc} \mu_\ell & \to & \mathbb{C}^\times \\ \omega_\ell & \mapsto & \zeta_\ell \end{array}$.

III - Type A & Combinatorics

Take $\lambda \vdash n$.

Take $\lambda \vdash n$.

γ_ℓ: the ℓ-core of λ [3]

(3). the partition obtained from λ by removing all hooks of length ℓ.

Take $\lambda \vdash n$.

γ_ℓ : the ℓ-core of λ [3]

g_ℓ : the size of γ_ℓ

(3). the partition obtained from λ by removing all hooks of length ℓ.

III - Type A & Combinatorics

Take $\lambda \vdash n$.

γ_ℓ : the ℓ-core of λ [3]

g_ℓ : the size of γ_ℓ

r_ℓ : the number of ℓ-hooks to remove from λ to obtain γ_ℓ $\quad (= \frac{n - g_\ell}{\ell})$

(3). the partition obtained from λ by removing all hooks of length ℓ.

Take $\lambda \vdash n$.

γ_ℓ : the ℓ-core of λ [3]

g_ℓ : the size of γ_ℓ

r_ℓ : the number of ℓ-hooks to remove from λ to obtain γ_ℓ $\quad (= \frac{n - g_\ell}{\ell})$

$w_{\ell,n} :=$

$$(\mathrm{g}_\ell + 1, \ldots, \mathrm{g}_\ell + \ell) \ldots (n - \ell + 1, \ldots, n) \in \mathfrak{S}_{r_\ell \ell}$$

(3). the partition obtained from λ by removing all hooks of length ℓ.

Take $\lambda \vdash n$.

γ_ℓ : the ℓ-core of λ [3]

g_ℓ : the size of γ_ℓ

r_ℓ : the number of ℓ-hooks to remove from λ to obtain γ_ℓ $\quad (= \frac{n - g_\ell}{\ell})$

$w_{\ell,n} :=$

$$(\mathrm{g}_\ell + 1, \ldots, \mathrm{g}_\ell + \ell) \ldots (n - \ell + 1, \ldots, n) \in \mathfrak{S}_{r_\ell \ell}$$

$C_{\ell,n} := \langle w_{\ell,n} \rangle$

(3). the partition obtained from λ by removing all hooks of length ℓ.

Take $\lambda \vdash n$.

γ_ℓ : the ℓ-core of λ [3]

g_ℓ : the size of γ_ℓ

r_ℓ : the number of ℓ-hooks to remove from λ to obtain γ_ℓ $\quad (= \frac{n - g_\ell}{\ell})$

$w_{\ell,n} :=$

$$(\mathrm{g}_\ell + 1, \dots, \mathrm{g}_\ell + \ell) \dots (n - \ell + 1, \dots, n) \in \mathfrak{S}_{r_\ell \ell}$$

$C_{\ell,n} := \langle w_{\ell,n} \rangle$ and $\theta_\ell : C_{\ell,n} \to \mathbb{C}^\times$ s.t. $\theta_\ell(w_{\ell,n}) = \zeta_\ell$.

---

(3). the partition obtained from λ by removing all hooks of length ℓ.

Take $\lambda \vdash n$.

γ_ℓ : the ℓ-core of λ [3]

g_ℓ : the size of γ_ℓ

r_ℓ : the number of ℓ-hooks to remove from λ to obtain γ_ℓ $\quad (= \frac{n - g_\ell}{\ell})$

$w_{\ell,n} :=$

$$(\mathrm{g}_\ell + 1, \dots, \mathrm{g}_\ell + \ell) \dots (n - \ell + 1, \dots, n) \in \mathfrak{S}_{r_\ell \ell}$$

$C_{\ell,n} := \langle w_{\ell,n} \rangle$ and $\theta_\ell : C_{\ell,n} \to \mathbb{C}^\times$ s.t. $\theta_\ell(w_{\ell,n}) = \zeta_\ell$.

$W_{\ell,n}^{g_\ell} := \mathfrak{S}_{g_\ell} \times C_{\ell,n} < \mathfrak{S}_n$

(3). the partition obtained from λ by removing all hooks of length ℓ.

If M is a $(\mathfrak{S}_n \times \mu_\ell)$-module,

$$M_i^\ell := \mathrm{Hom}_{\mu_\ell}(\tau_\ell^i, M), \quad \forall i \in [\![0, \ell - 1]\!]$$

If M is a $(\mathfrak{S}_n \times \mu_\ell)$-module,

$$M_i^\ell := \mathrm{Hom}_{\mu_\ell}(\tau_\ell^i, M), \quad \forall i \in [\![0, \ell - 1]\!]$$

Corollary

For each partition λ of n and each $i \in [\![0, \ell - 1]\!]$, one has the following equality :

$$\left[\left(\mathscr{P}^n_{|I_\lambda}\right)^\ell_i\right]_{\mathfrak{S}_n} = \sum_{j=0}^{\ell-1} \left[\mathrm{Ind}_{W^{\mathrm{g}_\ell}_{\ell,n}}^{\mathfrak{S}_n} \left(\left(\mathscr{P}^{\mathrm{g}_\ell}_{|I_{\gamma_\ell}}\right)^\ell_j \boxtimes \theta_\ell^{i-j}\right)\right]_{\mathfrak{S}_n}.$$

Thank you for your attention !

References

[Fo68] J. Fogarty. "Algebraic families on an algebraic surface". In : Amer. J. Math 90 (1968), p. 511-521.

[H03] M. Haiman. "Combinatorics, Symmetric functions, and Hilbert schemes". In : Current developments in mathematics (juill. 2003). doi : 10.4310/CDM.2002.v2002.n1.a2.