Rationality of finite groups: Groups with quadratic field of values

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Introduction

The concept of rationality associate to a finite group has always been an interesting topic both in representation theory and group theory.

Associating a field to any element $g \in G$ and having some global proprieties of those field can restrict a lot the structure of the group itself.

Notation

- Every group is finite.
- $x \sim y$ denotes the conjugation in the group.
- $|g|$ is the order of the element $g \in G$.
- $\mathbb{Q}_n := \mathbb{Q}(e^{2\pi i/n})$
- $Irr(G)$ denotes the set of irreducible complex characters of the group G.
- Cl_G denotes the set of conjugacy classes of G.

$n = \exp(G)$

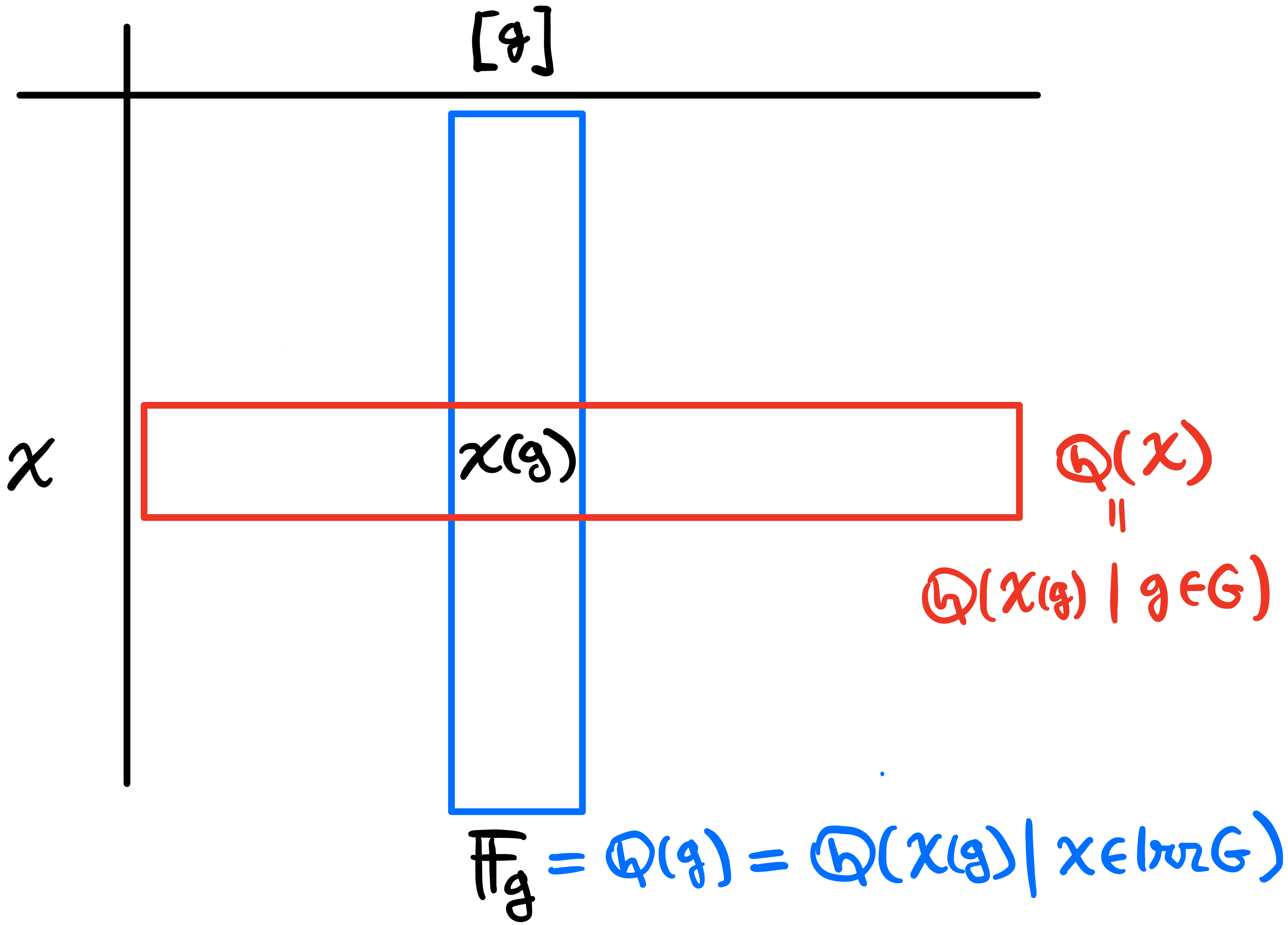
$$\mathrm{Gal}(\mathbb{Q}_n/\mathbb{Q}) \circlearrowleft \mathrm{Cl}_G$$

$$\sigma \qquad [g]^\sigma = [g^\pi]$$

$$\sigma(\zeta) = \zeta^\pi$$

$$\mathrm{Stab}_{\mathrm{Gal}(\mathbb{Q}_n/\mathbb{Q})}[g] = H \qquad \mathbb{F}_g := \mathrm{Fix}(H)$$

$\mathbb{F}_g$ is called <u>field of value of g</u>


[g]
χ
χ(g)
Q(χ) = Q(χ(g) | g∈G)
F_g = Q(g) = Q(χ(g) | χ∈Irr G)

Some definitions

Definition

A group G is called **quadratic rational** iff $\forall \chi \in \mathrm{Irr}(G)$ then $[\mathbb{Q}(\chi) : \mathbb{Q}] \leq 2$, where $\mathbb{Q}(\chi) = \mathbb{Q}(\chi(g) | g \in G)$.

Definition

An element $x \in G$ is called **semirational** iff its field of values $[\mathbb{Q}(x) : \mathbb{Q}] \leq 2$.
A group is called **semirational** iff every element is semirational.

Restrict $\mathbb{Q}_{|g|} \supseteq \mathbb{Q}(g)$

$|\mathbb{Q}(g) : \mathbb{Q}| \leq 2$

hence consider $\sigma \in Gal(\mathbb{Q}(g)/\mathbb{Q})$

$[g]^{\sigma} \neq [g]$

Some definitions

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Definition

An element x is called semirational if exists an integer $(m_x, |x|) = 1$ such that for any integer $(j, |x|) = 1$ then

$$x^j \sim x \text{ or } x^j \sim x^{m_x}$$

Some remarks

Constrain $\mathbb{Q}(g)$ ⟶ Constrain on $[g]$

m_g not UNIFORM

Definition

A group is called **r-semirational** if for any $x \in G$ $m_x = r$. In particular a group is called **inverse-semirational** if is $-1-$semirational.

A group is called UNIFORMLY SEMIRATIONAL (USR) if there exists an integer r $(r, \exp(G)) = 1$ st: G is r-semirational

Examples

- Rational groups, in particular S_n.
- $\mathbb{M}$ is inverse semirational.
- A_n is quadratic rational and semirational, in general not inverse-semirational.
- D_{10} is ± 3-semirational but not inverse semirational.
- `SmallGroup(32, 42)` is quadratic rational but not semirational.
- `SmallGroup(32, 9)` is semirational but not quadratic rational.

Cut equivalences

The previous family of groups is particularly interesting for the following proposition:

Proposition (Bächle, 2017)+(Ritter-Seghal, 1990)

The following are equivalent.

1. G is inverse semirational.
2. If $\mathbb{Q}G \cong \bigoplus_{k=1}^{m} M_{n_k}(D_k)$ is the Wedderburn decomposition where $m, n_k \in \mathbb{Z}_{\geq 1}$ and D_k rational division algebras for each k, then
$$\mathcal{Z}(D_k) \cong \mathbb{Q}(\sqrt{-d})$$
for some $d \in \mathbb{Z}_{\geq 0}$ square free.
3. G is quadratic rational and $\mathbb{Q}(\chi) \cap \mathbb{R} = \mathbb{Q}$ for any $\chi \in \mathrm{Irr}(G)$.

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4. G is **cut**.

Cut groups

In general we have the inclusion

$$\mathcal{Z}(\mathcal{U}(\mathbb{Z}G)) \geq \pm\mathcal{Z}(G)$$

but there is a family of groups that satisfies the following equality:

Definition

A finite group G is called **cut** (central units trivial) iff

$$\mathcal{Z}(\mathcal{U}(\mathbb{Z}G)) = \pm\mathcal{Z}(G)$$

Gruenberg-Kegel Graph (Prime graph)

Since a complete classification is a difficult task to accomplish, we can start to understand whenever those groups can be realizable by some special graph.
To every finite group G we can attach a graph that is related to the prime spectra of G.

Definition

The prime graph (or Gruenberg-Kegel graph) is the undirected loop-free and multiple-free graph whose vertices are the primes in the prime spectra of G, and two vertices p and q are joined by an edge, if and only if G contains an element of order pq.

Solvable case

Since we are interested in studying the Gruenberg-Kegel graph of those groups, would be nice to have a bound over the prime spectra mainly in the solvable case, let us denote $\pi(G) := \{p \mid p \mid |G|\}$

Theorem (Tent, 2012)

Let G be a solvable quadratic rational group. Then

$$\pi(G) \subseteq \{2, 3, 5, 7, 13\}$$

SHARP!

Theorem (Chilligan, Dolfi 2010-Bächle 2017)

Let G be a solvable semirational group. Then

$$\pi(G) \subseteq \{2, 3, 5, 7, 13, 17\}$$

?

If the group is inverse-semirational then

$$\pi(G) \subseteq \{2, 3, 5, 7\}$$

SHARP!

Frobenius quadratic rational groups

Corollary (MV)

Let G be a Frobenius group. Then the following are equivalent:

1. G is quadratic rational.
2. G is semirational.
3. G is USR.

USR groups

What about $\pi(G)$?

If $r^2 \equiv 1 \mod \exp(G)$ $\longleftrightarrow$ cut Groups

Theorem (MV)

Let G be a solvable group. Suppose that G is r-semirational with some r such that $r^2 \equiv 1 \mod n$. Then:

$$\pi(G) \subseteq \{2, 3, 5, 7\}$$

.

Equivalences of r-semirational groups

Proposition (MV)

Let G be a group with exponent n, $\mathbb{F}$ be a subfield of $\mathbb{Q}_n$ fixed by the cyclic subgroup generated by $\sigma_r \in \mathcal{G}al(\mathbb{Q}_n/\mathbb{Q})$ such that $(r, n) = 1$ and $\sigma_r(\zeta_n) = \zeta_n^r$. Then the following are equivalent:

1. G is r-semirational.
2. G is quadratic rational with $\forall \chi \in \mathrm{Irr}\, G$ st: $\mathbb{Q}(\chi) \cap \mathbb{F} = \mathbb{Q}$.

Which r makes a group r-semirational

We have seen that the same group can have different integer r such that G is r-semirational.

Definition

Let G be an USR group and $n = exp(G)$ then we call:

$$R_G := \{r \in \mathcal{U}(\mathbb{Z}/n\mathbb{Z}) | G \text{ is } r - semirational\}$$

We can observe that, fixed the group G, R_G is the coset of the group:

$$H_G = \{r \in \mathcal{U}(\mathbb{Z}/n\mathbb{Z}) | g^r \sim g \; \forall g \in G\} \cong \mathcal{G}al(\mathbb{Q}_n/\mathbb{Q}(G))$$

. And in particular

$$(\mathcal{U}(\mathbb{Z}/n\mathbb{Z}))^2 \leq H_G$$

Some questions

Question: What kind of R_G can appear?
In particular, can all possible cosets of compatible groups H_G appear?

Table: Possible R_G for quasi-rational 2-groups with exponent at least 8

$\{-1, 3\}$	$\{-1, -3\}$	$\{3, -3\}$	$\{-1\}$	$\{3\}$	$\{-3\}$
$\langle a \rangle_8 : \langle x \rangle_2$ $a^x = a^{-3}$	$\langle a \rangle_8 : \langle x \rangle_2$ $a^x = a^3$	$\langle a \rangle_8 : \langle x \rangle_2$ $a^x = a^{-1}$	$\langle a \rangle_8 \times \langle b \rangle_4 : \langle x \rangle_2$ $a^x = a^3$ $b^x = a^4 b^{-1}$ $a^y = a^5 b^2$ $b^y = b^{-1}$	$\langle a \rangle_8 \times \langle b \rangle_4 : \langle x, y \rangle_2$ $a^x = a^{-1}$ $b^x = a^4 b^{-1}$ $a^y = a^5 b^2$ $b^y = b^{-1}$	$\langle a \rangle_8 \times \langle b \rangle_4 : \langle x, y \rangle_2$ $a^x = a^{-1}$ $b^x = a^4 b^{-1}$ $a^y = a^5 b^2$ $b^y = a^4 b$

$\{2,3\}-$groups

$\{\pm 5, \pm 7\}$	$\{\pm 7, \pm 11\}$	$\{\pm 5, \pm 11\}$
$\langle a\rangle_{24} : \langle x, y\rangle_2$ $a^x = a^{-1}$ $a^y = a^{-11}$ SmallGroup(96,115)	$\langle a\rangle_{24} : \langle x, y\rangle_2$ $a^x = a^{-1}$ $a^y = a^{-5}$ SmallGroup(96,121)	$\langle a\rangle_{24} : \langle x, y\rangle_2$ $a^x = a^{-1}$ $a^y = a^{7}$ SmallGroup(96,117)
$\{-1, -7, 5, 11\}$	$\{-1, -11, 5, 7\}$	$\{-1, -5, 7, 11\}$
$\langle a\rangle_{24} : \langle x, y\rangle_2$ $a^x = a^{-11}$ $a^y = a^{-5}$ SmallGroup(96,183)	$\langle a\rangle_{24} : \langle x, y\rangle_2$ $a^x = a^{-5}$ $a^y = a^{11}$ SmallGroup(96,120)	$\langle a\rangle_{24} : \langle x, y\rangle_2$ $a^x = a^{5}$ $a^y = a^{-11}$ SmallGroup(96,113)
	$\{-1, -11, -5, -7\}$	
	$\langle a\rangle_{24} : \langle x, y\rangle_2$ $a^x = a^{5}$ $a^y = a^{11}$ SmallGroup(96,118)	
$\{-1, 11\}$	$\{7, -5\}$	$\{7, 11\}$
SmallGroup(192,95)	SmallGroup(192,305)	SmallGroup(192,412)
$\{5, 7\}$	$\{-1, -7\}$	$\{\pm 7\}$
SmallGroup(192,414)	SmallGroup(192,713)	SmallGroup(192,415)
$\{-1, 7\}$	$\{-7, -5\}$	$\{5, -7\}$
SmallGroup(192,418)	SmallGroup(192,435)	SmallGroup(192,623)
$\{-1, -5\}$	$\{\pm 5\}$	$\{11, -5\}$
SmallGroup(192,440)	SmallGroup(192,949)	SmallGroup(192,438)
$\{-1, 5\}$	$\{5, 11\}$	$\{11, -7\}$
SmallGroup(192,1396)	SmallGroup(192,632)	SmallGroup(192,726)
$\{7\}$	$\{-5\}$	$\{-1\}$
SmallGroup(192,424)	SmallGroup(192,445)	SmallGroup(192,634)
$\{5\}$	$\{11\}$	$\{-11\}$
SmallGroup(192,595)	SmallGroup(192,631)	?

In the previous situation we have fixed an exponent but we can observe that the situation seems indipendent to the choice of the exponent but more like on the prime spectra of the group. In fact H_G can be viewed as a subgroup of $\mathcal{U}(\mathbb{Z}/n\mathbb{Z})/\mathcal{U}(\mathbb{Z}/n\mathbb{Z})^2$.

Question: Fix a coset R_G that is realized by some group of exponent n. Suppose to have another m such that

$$\mathcal{U}(\mathbb{Z}/n\mathbb{Z})/\mathcal{U}(\mathbb{Z}/n\mathbb{Z})^2 \cong \mathcal{U}(\mathbb{Z}/m\mathbb{Z})/\mathcal{U}(\mathbb{Z}/m\mathbb{Z})^2$$

Can we find another group H of exponent m such that $R_H = R_G$?

Theorem (delRio, MV)

Let $p \in \{2, 3\}$ and R any coset as before. Then for any n positive integer then exists a group G of exponent p^n that is USR and realizes R, meaning $R_G = R$.


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Thank you