Division and Localization on Groupoid Graded Rings

Caio Antony Gomes de Matos Andrade

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Motivation

- Field of fractions: embedings $R \to K$, K division ring generated as a division ring by the image of R.
- Commutative case: $S \subseteq R \mapsto S^{-1}R$
- Noncommutative case: Ore domains
- Noncommutative case: maps which invert square matrices (Cohn, 1971).
- Was generalized to group graded rings (Kawai, Sánchez)

Goal:

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Let’s start with some drawings!

Groupoids

Definition

A groupoid Γ is a small category in which every arrow is an inversible.

- Groupoid $=$ arrows of Γ;
- $\Gamma_0 =$ objects of $\Gamma =$ idempotents of Γ;
- Groupoids as semigroups;

Example

I set, then $\Gamma = I \times I$ is a groupoid

$$(i,j)(j,k) = (i,k).$$

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Groupoid Graded Rings

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Γ *groupoid, R ring. We say that R is a* Γ***-graded ring*** *if*

1. $R = \oplus_{\gamma \in \Gamma} R_\gamma$, *$R_\gamma$ additive subgroup;*
2. *If $\gamma\delta$ exists, then* $R_\gamma R_\delta \subseteq R_{\gamma\delta}$*;*
3. *If $\gamma\delta$ doesn't exist, then* $R_\gamma R_\delta = 0$*;*
4. *(Cala, Lundström, Pinedo 2021) For every $e \in \Gamma_0$, there exists idempotents $1_e \in R_e$ such that, for every $t(\gamma) \xleftarrow{\gamma} d(\gamma) \in \Gamma$, $x \in R_\gamma$, we have*

$$1_{t(\gamma)} x = x = x 1_{d(\gamma)}.$$

Definition

Idempotent support of** R**: $\Gamma_0(R) = \{e \in \Gamma_0 : 1_e \neq 0\}$.

We can suppose that $\Gamma_0(R) = \Gamma_0$ by taking the full subgroupoid of Γ whose objects are $\Gamma_0(R)$.

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Division on Groupoid Graded Rings

Definition

(Graded) invertible element: *for $x \in R_\gamma \setminus \{0\}$, there exists $y \in R_{\gamma^{-1}}$ such that $xy = 1_{t(\gamma)}$ and $yx = 1_{d(\gamma)}$.*

Example

$\Gamma = \{e_1\} \cup \{e_2\}$, $R = R_{e_1} \times R_{e_2}$, *where* $R_{e_1} = R_{e_2} = \mathbb{Q}$. *Then, R is a Γ-graded division ring, which has proper graded ideals $R_{e_1} \times \{0\}$ and $\{0\} \times R_{e_2}$.*

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Maps between groupoid graded rings

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Let R, S be Γ-graded rings. We say that a ring homomorphism homomorphism $f : R \longrightarrow S$ is a ***graded ring homomorphism*** *if*

1. $f(R_\gamma) \subseteq S_\gamma$, *for all* $\gamma \in \Gamma$;
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Homogeneous Matrices

Definition

Homogeneous Matrices: $\overline{\alpha} \in \Gamma^m, \overline{\beta} \in \Gamma^n$. *Consider*

$$M_{\overline{\alpha} \times \overline{\beta}}(R) = \left\{ A \in \begin{pmatrix} R_{\alpha_1 \beta_1^{-1}} & \dots & R_{\alpha_1 \beta_n^{-1}} \\ \vdots & \ddots & \vdots \\ R_{\alpha_m \beta_1^{-1}} & \dots & R_{\alpha_m \beta_n^{-1}} \end{pmatrix} \right\}$$

where $R_{\alpha_i \beta_j^{-1}} = 0$ *if* $\alpha_i \beta_j^{-1}$ *is not defined.*

Set of all homogeneous matrices: $\mathcal{M}(R)$.

$$I_{\overline{\alpha}} = \begin{pmatrix} 1_{t(\alpha_1)} & 0 & \dots & 0 \\ 0 & 1_{t(\alpha_2)} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1_{t(\alpha_n)} \end{pmatrix}, \qquad I_{\overline{\alpha}} A = A I_{\overline{\beta}} = A, \text{ for } A \in M_{\overline{\alpha} \times \overline{\beta}}(R)$$

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Definition

Let $\overline{\alpha}, \overline{\beta} \in \Gamma^n$. We say that $A \in M_{\overline{\alpha}\times\overline{\beta}}(R)$ is ***invertible*** *if there exists $B \in M_{\overline{\beta}\times\overline{\alpha}}(R)$ such that*

$$AB = I_{\overline{\alpha}}, \qquad BA = I_{\overline{\beta}}.$$

We're now ready to attack our problem!

Goal:

Study epimorphisms from a given groupoid graded ring to a division groupoid graded ring by means of the homogeneous matrices which are mapped to invertible matrices.

Definition

Let $f : R \longrightarrow S$ be a graded ring homomorphism and $\Sigma \in \mathcal{M}(R)$ be such that $f(A)$ is invertible for every $A \in \Sigma$. We call f a Σ-inverting (graded ring) homomorphism.

Definition

Let R be a Γ-graded ring and $\Omega \subseteq \Gamma_0(R)$. We say that $\Sigma \subseteq \mathcal{M}_\Omega(R)$ is ***graded (lower) semimultiplicative*** *if*

1. $(1_e) \in \Sigma$, *for every* $e \in \Gamma_0(R)$;
2. *If* $A \in \Sigma \cap M_{\overline{\alpha} \times \overline{\beta}}(R), B \in \Sigma \cap M_{\overline{\alpha'} \times \overline{\beta'}}(R)$, *then*

$$\begin{pmatrix} A & 0 \\ C & B \end{pmatrix} \in \Sigma$$

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Graded semimultiplicative sets show up naturally.

Proposition

Let $f : R \longrightarrow S$ *a graded ring homomorphism such that* $\Gamma_0(R) = \Gamma_0(S)$. *Then, the set*

$$\Sigma = \{A \in \mathcal{M}(R) : f(A) \text{ is invertible in } S\}$$

is graded semimultiplicative.

Graded Rational Closure

Warning: Technical definition, but don't worry!

Definition

Let $\Sigma \in \mathcal{M}(R)$ and $f : R \longrightarrow S$ be a Σ-inverting homomorphism. For $\gamma \in \Gamma$, we define the ***homogeneous rational closure of degree*** γ *as the set $(R_f(\Sigma))_\gamma$ consisting of all $x \in S$ such that there exists $\overline{\alpha}, \overline{\beta} \in \Gamma^n, A \in \Sigma_{\overline{\alpha}\times\overline{\beta}}$ such that $\gamma = (\alpha_i\beta_j^{-1})^{-1} = \beta_j\alpha_i^{-1}$, and x is the (j,i)-th entry of $(A^f)^{-1}$. The* ***graded rational closure****, denoted by $R_f(\Sigma)$, is the additive subgroup of S generated by $\bigcup_{\gamma\in\Gamma}(R_f(\Sigma))_\gamma$.*

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Theorem

Let $f : R \longrightarrow S$ be a graded ring homomorphism such that $\Gamma_0(R) = \Gamma_0(K)$. Set

$$\Sigma = \{A \in \mathcal{M} : A^f \text{ is invertible over } S\}.$$

Then, $R_f(\Sigma)$ is a Γ-graded ring. Furthermore, if $x \in (Q_f(\Sigma))_\gamma$ is invertible in S, then $x^{-1} \in (Q_f(\Sigma))_{\gamma^{-1}}$. Thus, if S is a Γ-graded division ring, then $R_f(\Sigma)$ is a Γ-graded division subring of S.

Theorem (CA, del Río, Sánchez)

Let $f : R \longrightarrow S$ be a Γ-graded ring homomorphism such that $\Gamma_0(R) = \Gamma_0(K)$, Σ be a graded lower semimultiplicative subset of $\mathcal{M}(R)$ such that f is Σ-inverting. Then, the map $f : R \longrightarrow R_f(\Sigma)$ is an epimorphism of Γ-graded rings.

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Graded Universal Localization

Definition

*(R, Σ)-**INV:** Category of Σ-inverting homomorphisms.*

- *Objects: Σ-inverting homomorphisms $f : R \to S$*
- *Arrows: Graded ring homomorphisms $S \to S'$ such that*

$$\begin{array}{ccc} R & \longrightarrow & S \\ & \searrow & \downarrow \\ & & S'. \end{array}$$

Definition

*A **universal localization** of R at Σ is a an initial object in the category (R, Σ)-**INV**.*

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Proposition

Let R be a Γ-graded ring and $\Sigma \subseteq \mathcal{M}(R)$. Then, the following statements hold true. There exists a universal localization $\lambda : R \to R_\Sigma$ of R at Σ, and the map λ is a graded ring epimorphism.

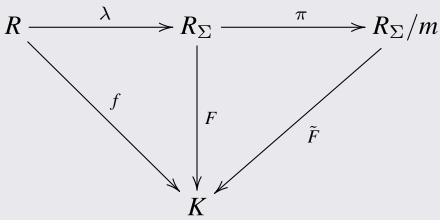
Theorem (CA, del Río, Sánchez)

Let R be a Γ-graded ring, K be a Γ-graded division ring and $f : R \to K$ be a epimorphism of Γ-graded rings such that $\Gamma_0(R) = \Gamma_0(K)$. Let

$$\Sigma = \{A \in \mathcal{M}(R) : f(A) \text{ is invertible over } K\},$$

$$m = \langle x \in R : x \text{ homogeneous and not invertible}\rangle,$$

we have that $R_\Sigma/m \simeq R_f(\Sigma)$, $\pi\lambda : R \to R_\Sigma/m$ is an epimorphism and there exists an isomorphism of Γ-graded R-rings $\tilde{F} : R_\Sigma/m \to K$ such that the following diagram is commutative.



What now?

- When is R_Σ actually an interesting ring? (Known as Malcomsom's Criterion)
- For which $\Sigma \subseteq \mathcal{M}(R)$ is R_Σ "graded local"and $R_f(\Sigma)$ a division ring? (scarily technical in the simpler cases)

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Grazie mille!!!

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