On the Isomorphism Problem for group rings

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Glimmers of hope

We remark that it is really not surprising that $K[G_1]$ and $K[G_2]$ are isomorphic for fields of characteristic $\neq q$. What is surprising is that they are isomorphic when char $K = q$ and that this isomorphism is so easily proved. In any case, we see that the group rings over all fields do not contain sufficient information to uniquely determine the group. There are, however, two glimmers of hope. The first concerns integral group rings, and the second concerns p-groups over $GF(p)$. The remainder of this section is devoted to a brief discussion of these two topics. In particular, we show that for the Dade examples we have $Z[G_1] \not\cong Z[G_2]$.

IP($\mathbb{Z}$) holds for

- Abelian. G. Higman, 1940
- Metabelian. A. Whitcomb, 1968.
- p-groups, K. W. Roggenkamp, L. Scott, 1987. Also IP($\mathbb{Z}_p$).
- Nilpotent groups, A. Weiss, 1991.
- Abelian-by-nilpotent, K. W. Roggenkamp, L. Scott, 1991.
- Nilpotent-by-abelian, Kimmerle,
- Supersolvable, Kimmerle, 1991.
- Frobenius and 2-Frobenius, Kimmerle 1991.

Positive results about MIP

- Abelian, W. E. Deskins 1956. group algebra),
- Groups with center of index p^2, V. Drensky 1989.
- Metacyclic groups, C. Bagiński 1988, R. Sandling 1996.
- $D_3(G) = 1$, I. B. S. Passi and S. Sehgal 1972.
- $D_4(G) = 1$ and $p \neq 2$, M. Hertweck 2007.
- Groups containing a cyclic subgroup of index p^2, C. Bagiński and A. Konovalov 2007.
- $|G| \mid 2^8$, D. S. Passman 1965, Wursthon 1993, M. Hertweck and M., Soriano, 2007, B. Eick and A. Konovalov 2011, L. Margolis and T. Moede, 2020.
- $|G| \mid p^5$, $p \neq 2$, Makasikis 1976, M. A. M. Salim and R. Sandling 1996.
- 2-generated of class 2, O. Broche, dR, 2021.
- 2-groups with cyclic center such that $G/Z(G)$ is dihedral [84], L. Margolis, T. Sakurai, M. Stanojkovski 2023.
- 2-groups of class 3 such that $[G : Z(G)] = |\Phi(G)| = 8$, L. Margolis, T. Sakurai, M. Stanojkovski 2023.

Counter-example to MIP(2). García-Lucas,Margolis,dR, 2021

$n > m > 2$

$$\left.\begin{array}{c} G \\ H \end{array}\right\} = \left\langle a, b, c \;\middle|\; \begin{array}{c} c = [b, a], \\ a^{2^n} = b^{2^m} = c^4 = 1, \\ c^a = c^{-1} \end{array}, c^b = \begin{cases} c^{-1} \\ c \end{cases} \right\rangle.$$

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Margolis,Sakurai, 2024

The 2-generated counter-examples to MIP(2) with dihedral central quotient are precisely

$$\left.\begin{array}{c} G \\ H \end{array}\right\} = \left\langle a, b, c \;\middle|\; \begin{array}{c} c = [b, a], \\ a^{2^n} = b^{2^m} = c^{2^k} = 1, \\ c^a = c^{-1} \end{array} , c^b = \begin{cases} c^{-1} \\ c \end{cases} \right\rangle.$$

with $n > m > k \geq 2$.

Comparing the counterexample with known results

- Groups with center of index p^2, V. Drensky 1989.
- $|G| \mid 2^8$, D. S. Passman 1965, Wursthon 1993, M. Hertweck and M., Soriano, 2007, B. Eick and A. Konovalov 2011, L. Margolis and T. Möede, 2020.
- 2-generated of class 2, O. Broche, dR, 2021.
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- $C_G(G')/G' \ncong C_H(H')/H'$.
- $C_G(G')/C_G(G')' \ncong C_H(H')/C_H(H')'$.
- $G/\gamma_3(G) \ncong H/\gamma_3(H)$.
- $G/(G')^{2^3} \ncong H/(H')^{2^3}$.

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García-Lucas, Stanojkovski, dR, 2023

$\mathrm{char}(F) = p > 2$. Let G and H be 2-generated p-groups with cyclic derived subgroup. If $FG \cong FH$ then

- $Exp(C_G(G')) = Exp(C_H(H'))$.
- $C_G(G')/G' \cong C_H(H')/H'$.
- $C_G(G')/C_G(G')' \cong C_H(H')/C_H(H')'$.
- $G/\gamma_3(G) \cong H/\gamma_3(H)$.
- $G/(G')^{p^3} \cong H/(H')^{p^3}$.

Thanks!