Endotrivial complexes

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Conventions & Notation

- G is a finite group.
- k is a field of characteristic $p > 0$.
- $s_p(G)$ denotes the set of p-subgroups of G.
- $\mathrm{Syl}_p(G)$ denotes the set of Sylow p-subgroups of G.
- All kG-modules are finitely generated.
- ${}_{kG}\mathbf{triv}$ is the category of f.g. p-permutation kG-modules.

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- The group $\mathcal{E}_k(G)$ of endotrivial complexes forms a rational p-biset functor, and is the Picard group of $K^b({}_{kG}\mathbf{triv})$.

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A main result: We have classified all endotrivial complexes!

Motivation: endotrivial modules

A kG-module M is endotrivial if $M^* \otimes_k M \cong k \oplus P$, for some projective module P, i.e. $M^* \otimes_k M \cong k \in \text{stmod}(kG)$. These are the invertible objects of $\text{stmod}(kG)$.

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$$T_k(G) := \{[M] \in \text{stmod}(kG) \mid M \text{ is endotrivial}\}.$$

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Known results

- $T_k(G)$ is finitely generated abelian. (Puig '90, CMN '06)
- $T_k(G)$ is determined for p-groups. (CT '00-'05)
- $T_k(G)$ is determined for somme finite groups of Lie type (CMN '06)
- ...and many, many more!

Determining $T_k(G)$ for all groups remains open.

Preliminaries

Definition (p-permutation)

A kG-module M is a

- permutation module if $M \cong k[X]$ for some G-set X.
- p-permutation module if for $S \in \mathrm{Syl}_p(G)$, $\mathrm{res}^G_S M$ is a permutation module, or equivalently, if M is a direct summand of a permutation module.

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Definition (Brauer construction)

- Given $P \in s_p(G)$, the Brauer construction is an additive functor $-(P) : {}_{kG}\mathbf{triv} \to {}_{k[N_G(P)/P]}\mathbf{triv}$.
- For $M, N \in {}_{kG}\mathbf{triv}$, we have a natural isomorphism

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Think of the Brauer construction as a "modular fixed points" functor. Indeed, $k[X](P) \cong k[X^P]$.

Endotrivial complexes

Definition

- A bounded chain complex $C \in Ch^b({}_{kG}\mathbf{triv})$ is endotrivial if

$$\mathrm{End}_k(C) \cong C^* \otimes_k C \simeq k[0].$$

i.e. $C^* \otimes_k C \cong k[0] \oplus D$ for some contractible chain complex D. C is an invertible object of $K^b({}_{kG}\mathbf{triv})$.

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- Let $\mathcal{E}_k(G)$ denote the set of homotopy classes of endotrivial kG-complexes. $(\mathcal{E}_k(G), \otimes_k)$ forms an abelian group, and is the Picard group of $K^b({}_{kG}\mathbf{triv})$.

Examples

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2. Let $n \geq 3$ and let H_1, H_2 be noncentral, nonconjugate subgroups of D_{2^n} of order 2.

$$\begin{array}{ccccc} & & k[D_{2^n}/H_1] & & \\ & \nearrow\!\!\!\!\nearrow & & \searrow\!\!\!\!\searrow & \\ kD_{2^n} & & \oplus & & k \\ & \searrow\!\!\!\!\searrow & & \nearrow\!\!\!\!\nearrow & \\ & & k[D_{2^n}/H_2] & & \end{array}$$

The homomorphisms are induced from G-set homomorphisms $G/H \to G/K, gH \mapsto gK$.

Splendid Rickard equivalences

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Theorem

Let C be an endotrivial complex of kG-modules. Let $\phi \in \mathrm{Aut}(G)$ and set

$$\Delta_\phi G = \{(\phi(g), g) \in G \times G \mid g \in G\} \cong G.$$

$\mathrm{ind}_{\Delta_\phi G}^{G\times G} C$*, regarded as a chain complex of* (kG, kG)*-bimodules, is a splendid Rickard autoequivalence of* kG*.*

Orthogonal units

The trivial source ring $T(kG)$ is the Grothendieck group of ${}_{kG}\mathbf{triv}$.

$$O(T(kG)) = \{u \in T(kG)^{\times} : u^{-1} = u^{*}\}.$$

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Let C be an endotrivial complex.

$$\Lambda(C) = \sum_{i \in \mathbb{Z}} (-1)^{i} [C_i] \in O(T(kG)),$$

and $\Lambda : \mathcal{E}_k(G) \to O(T(kG))$ is a well defined group homomorphism.

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Conjecture: $\Lambda : \mathcal{E}_{\mathbb{F}_p}(G) \to O(T(\mathbb{F}_p G))$ is surjective.

Homology

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Theorem

Let $C \in Ch^b({}_{kG}\mathbf{triv})$. The following are equivalent:

- *C is endotrivial.*
- *For every $P \in s_P(G)$, $C(P)$ has nonzero homology in exactly one degree, and that homology has k-dimension 1. That is, $C(P)$ is an invertible object in $D^b({}_{kG}\mathbf{triv})$.*

h-marks

Definition

- If C is endotrivial and $P \in s_p(G)$, let $h_C(P)$ denote the degree in which $C(P)$ has nontrivial homology. Say $h_C(P)$ is the h-mark of C at P.

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Question: How much do "local" homological properties, like the h-marks, determine the structure of an endotrivial complex?

Answer: Almost entirely!

The h-mark homomorphism

Theorem

$$h : \mathcal{E}_k(G) \to C(G,p)$$
$$[C] \mapsto h_C$$

is a well-defined group homomorphism, with $\ker h \cong \mathrm{Hom}(G, k^\times)$*, the torsion subgroup of* $\mathcal{E}_k(G)$.

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In particular, $\mathcal{E}_k(G)$ is finitely generated with $\mathbb{Z}$-rank bounded by the number of conjugacy classes of p-subgroups of G. If G is a p-group, h is injective.

We call h the h-mark homomorphism.

Results for p-groups

The group of class functions $C(G, p)$ has a subgroup $C_b(G, p)$, the subgroup of Borel-Smith functions. These arise from homotopy representations of the sphere, and as the kernel of the Bouc homomorphism.

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2. *We may assign rational* p*-biset functor structure to* $\mathcal{E}_k$ *via transport. Restriction, inflation, and deflation are all what we expect, but induction is* **not** *tensor induction.*
3. $\Lambda : \mathcal{E}_k(G) \to O(T(kG))$ *is surjective.*
4. *Given any* p*-permutation autoequivalence* γ *of* kG*, there exists a splendid Rickard autoequivalence* X *of* kG *for which* $\Lambda(X) = \gamma$*.*

Results for non-p-groups

Theorem

Let G be a finite group and $S \in \mathrm{Syl}_p(G)$.

$$\mathrm{res}^G_S : \mathcal{E}_k(G) \to \mathcal{E}_k(S)$$

has image $\mathcal{E}_k(S)^{\mathcal{F}} \le \mathcal{E}_k(S)$, the fusion-stable subgroup of $\mathcal{E}_k(S)$ which consists of elements $[C] \in \mathcal{E}_k(S)$ for which $h_C(P) = h_C(Q)$ for all G-conjugate $P, Q \le S$.

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Corollary

$$\mathcal{E}_k(G) \cong \mathcal{E}_k(S)^{\mathcal{F}} \times \mathrm{Hom}(G, k^\times) \cong C_b(G,p) \times \mathrm{Hom}(G, k^\times).$$

Thank you!!

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